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Analysis and solutions to the stable roommate problem: An examination of complexity, stability, and matching algorithms

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Abstract

Roommate issues with strong preferences are defined here. An extension of the Stable Marriage Problem that does not include two partners is the Stable Roommates Problem, or SR. The complexity and approximability of the problem of computing an egalitarian stable matching in instances of d-SRI.

Keywords: Approximability, problem, egalitarian, marriage, stability

Introduction

Economic literature has mostly modelled the marriage market as a bipartite matching game with transferable utility (TU) since Becker's fundamental work (1973, 1991). There are two people in a pair, and they come from different parts of society; we'll refer to them as men and women, respectively. All possible couples in the TU model have an excess that they (endogenously) divide up between themselves. To be stable, the resultant matching must be resistant to both one-way and two-way variations. Koopmans and Beckmann presented the theoretical study of bipartite matching with transferable utility (1957). Stable matchings maximize aggregate surplus, and the related individual surpluses solve the dual imputation issue; this set and demonstrated that it is both the core and the set of competitive equilibria under TU.

In contrast to the stable marriage problem, which requires bipartition, the stable roommate problem allows any two vertices to be matched.

The set of all finite cardinalities is denoted by F in Definition 5.1. After that, $(F, (<_f)_{f \in F})$ is an instance of the stable roommate problem (SRP), where each $<_f$ is a total ordering of some set $F_f \subseteq F \setminus \{f\}.$ Additionally, G, a spanning subgraph of where V (G) = F is the underlying graph or related graph for the SRP instance. Here in graph G we can see all the possible matching pairings.

Continuing with the terminology from earlier chapters, we may see the total orderings as rankings of potential roommate candidates. We may presume, using the logic of Chapter 3's reasoning, that $G = K_{|F|}, |F|$ every preference list is filled, and it's even. From here on out, we'll talk about the stable roommate issue under these assumptions. The notion of a stable marriage is identical to that of a steady roommate arrangement.

Definition: 1. For an instance $(F, (<_f)_{f \in F})$ A matching A ∈ E(G) is stable in the context of SMP and the related graph G in the absence of an edge. $f_1 f_2 \in G \setminus A$ in which the following two conditions hold:

f_1 is unmatched in A, or $f_2 <_{f_1} (f_1)_A$

f_2 is unmatched in A, or $f_1 <_{f_2} (f_2)_A$

An SRP instance may not have a stable assignment, unlike the stable marriage problem. Take a collection of four individuals {1, 2, 3, 4} as an example. In this set, person 4 is the final option for all other individuals, and for each i in {1, 2, 3}, person i+ 1 (mod 3) is the person that person most prefers. So, in every perfect matching, there is an i from the set {1, 2, 3} that is paired with 4, and the other two individuals are paired with each other. The matching cannot be stable since i−1 (mod 3) would choose i and i would choose i − 1 over 4.

Using the example provided by Gusfield (1988) [15] in

Figure1, we shall demonstrate the findings. Furthermore, it is assumed while addressing the roommate dilemma generally that $(F, (<_f)_{f \in F})$ unless explicitly said otherwise, is the case of SRP.

1	7	2	6	8	5	3	4
2	4	6	5	3	8	1	7
3	5	2	1	7	4	6	8
4	1	7	3	6	5	8	2
5	7	1	8	4	6	2	3
6	7	3	8	4	5	1	2
7	2	8	4	3	5	6	1
8	4	2	3	5	6	7	1

Fig 1: (Gus field 1988) [15] Preference lists in the running example

Using the technique in (Irving 1985) [14] that discovers a stable roommate assignment if one exists and declares that no stable assignment exists otherwise, we may first demonstrate the generalization of rotations. We shall adhere to the terminology laid forth by Gusfield (1988) [15]. First, second, and last will be used in the same way as in the preceding chapters. On occasion, we'll say that the first item in a list is the head of the list. Recalling from earlier chapters, we will also use the symbol rf (g) to indicate g's position on f's list.

Literature Review

Mertens, Stephan. (2015) [5]. There is a time and space complexity of $O(n^2)$ for the generic stable roommates issue with n agents. However, random examples may be solved more quickly and using less memory. On average, the time and space complexity of our approach is $O(n^\frac{3}{2})$ under randomized circumstances. With this approach, we may model the stable roommate's issue on a large scale and find the probability p_n that a random n-size case allows a stable matching. Our findings lend credence to the hypothesis that $p_n = \Theta(n^{-1/4})$.

FIDAN, MÜGE & ERDEM, ESRA. (2021). Agents' preferences for potential roommates, which may include ties or be incomplete, constitute the Stable Roommates problem with Ties and Incomplete lists (SRTI), a kind of matching issue. Stable matching is what SRTI is aiming for, and it may optimize a fairness metric that is not specific to any one domain (e.g. Egalitarian). On the other hand, when it comes to practical applications, like choosing dorm roommates, we often take into account a range of domain-specific factors that are based on preferences rather than the agents' habits and wishes. In light of this, we provide a knowledge-based approach to SRTI that takes domain-specific information into account and explore its practical use in university dorm roommate assignment.

Contreras, José & Torres-Martínez, Juan. (2021). This work takes the flatmate dilemma a step further by introducing externalities, which make it possible for partner preferences to be situationally dependent. We present and characterize stability notions for sets of agents that may be partitioned or matched, assuming that all agents can reasonably anticipate how other agents will respond to outliers. We demonstrate that a stable partition exists for any flatmate issue including externalities, and that a stable matching is present in the absence of odd rings in such a partition. Based on these findings, we may identify constraints on the preference space that guarantee a stable matching exists. Furthermore, we demonstrate that when externalities are included, some classical features are forfeited, such as the core's coincidence with the set of stable matchings, the invariance of the set of agents who are alone in a stable matching, and the availability of pathways to stability from any unstable matching.

Kavitha, Telikepalli. (2018) [3]. In this example of roommates with stringent preference lists, we examine the well-known matching issue. In a bipartite instance, popular matchings are guaranteed to be present; however, in a roommate's instance, they are not. We prove, after years of speculation, that the well-known flatmate matching issue is NP-hard in its complexity. There has been much research on dominant matchings, a subclass of max-size popular matchings in bipartite graphs. Even when the instance allows a stable matching, we prove that the dominating matching issue in a housemate's instance is NP-hard.

Logapriya, N. (2024) [4]. When looking for a flatmate match, the Satisfactory Roommates Problem (SFRP) is a good place to start. Everyone in the set with an even cardinality of 2n ranks the other 2n-1 members in preference according to the SFRP. Based on each person's degree of satisfaction, the set is divided into 2n/2 pairs of roommates using the satisfactory matching method. There are three people in this Three-individual Satisfaction-Roommate Problem (TPSRP), and each individual will be given a list of two potential roommates. Each member of the set might have their choice ranked with the other (3n-1) members. A collection of triples is called a matching. In order to locate rooms with perfect triples, this article presents a newly developed approach.

Generalization to the roommate problem

Definition 1: Person f is considered semi-engaged to first(f) at any stage in the algorithm if and only if last(first(f)) = f. Everybody who isn't halfway committed is at liberty.

Algorithm 1: (First Stage): Go through the motions again:

1. The algorithm should be terminated since no reliable assignment can be made if the list is empty.
2. Wrap up phase 1 if most people are interested.
3. Pick a free person f and erase k from first(f)'s list and first(f) from k's list for every person k who is rated lower than f on first(f)'s list. After that, f gets somewhat involved with first(f).

Definition 2: The phase 1 table is the collection of lists that are generated after phase 1 is executed.

Figure 2 displays the phase 1 table for our scenario. Details on the building may be found in Appendix A.

1	2	6	5	3	4		
2	6	5	3	8	1		
3	5	2	1	7	4	6	
4	1	7	3	6	5	8	
5	7	1	8	4	6	2	3
6	3	8	4	5	1	2	
7	8	4	3	5			
8	4	2	5	6	7		

Fig 2: (Gus field 1988) [15] Phase 1 table in the running example

There are no stable assignments that pair i to j in the phase 1 table, as Irving (1985) [14] demonstrates, if j is absent from i's list. The stable marriage problem's male-optimal shortlists are similar to the phase 1 table in this regard. Here, the evidence is shown.

Proposition 1: (Arthur 1985) If i does not have j on its list, then there are no stable assignments in the phase 1 table that pair i with j.

Proof: By calculating the number of iterations in phase 1, we may demonstrate this by induction. The outcome is meaningless if there are no iterations. Assume for a moment that there exist no stable assignments that link i with j if, after n−1 (n ≥ 1) iterations, j is not included in i's list.
So, let's pretend that we've gone through n rounds of phase 1, but i forgot to include j in his list. Assume that j gets eliminated from i's list in then'th iteration if this occurred before then'th iteration; else, we are done with induction. The removal of i from j's list occurs before the removal of j from i's list, and we may generalize from there. Assuming j = first(k), we can deduce that i gets removed from j's list the moment a person k becomes semi-engaged to j. and $k <_j i.$
Assume that ij is an element of a stable assignment A. Next, we may deduce by induction that since j is equal to first(k) and no one who is not on j's list after n − 1 iterations can be matched with j, we can conclude that $j <_k k_A.$ Therefore, there is an argument against A's stability.
In order to avoid instability, we shall always assume that the phase 1 database never contains empty lists. Presenting rotations is the next step. The definitions used in the stable marriage issue should serve as a model for the following. High is equal to first(ei) and si is equal to second(ei) if and only if ei is a member of F.

Definition 3: A disclosing rotation in a given table T, R is a sorted subset of the individuals represented by E = (e0, e1..., er−1), where si = hi+1 for all 0 ≤ i < r, with i + 1 chosen mod r. A common notation for this is R = (E, H, S), where H is the collection of first elements in E ordered according to E and S is the collection of second entries ordered similarly. To make notation easier, we utilize both H and S, even though H dictates S.

Definition 4. Let $R_1 = (E_1, H_1, S_1)$ and $R_2 = (E_2, H_2, S_2)$ be rotations, and let $E = ((e_1)_0, (e_1)_1, \ldots, (e_1)_{r-1})$ and $E_2 = ((e_2)_0, (e_2)_1, \ldots, (e_2)_{r-1})$. If there is an integer, then R1 and R2 are equal. $k\ (0 \leq k \leq r-1)$ such that $(e_1)_i = (e_2)_{i+k}$, $(h_1)_i = (h_2)_{i+k}$, and $(s_1)_i = (s_2)_{i+k}$ for all $0 \leq i < r$, where the mod r calculation is performed. The fact that this connection is an equivalence means that $R_1 = R_2.$

Definition 5: To erase an exposed rotation R = (E, H, S), we first remove every element below ei from si's list for every si ∈ S. Then, for each k removed from si's list, we remove si from k's list.
The outcome of removing a rotation from the running example's phase 1 table is seen in Figure 3. The eliminated rotation is denoted as R = (E, H, S), with E = {1, 2, 3}, H = {2, 6, 5}, and S = {6, 5, 2}. For information, go to Appendix A. The second and last stage of the algorithm will be shown after this.

1	6	5	3	4		
2	5	3				
3	2	1	7	4	6	
4	1	7	3	6	5	8
5	7	1	8	4	6	2
6	3	8	4	5	1	
7	8	4	3	5		
8	4	5	6	7		

Fig 3: (Gus field 1988) [15] The result of eliminating the rotation R from the phase 1 table

The stable roommates' problem with ties
For every particular SR instance, all 2n participants are asked to rate the other participants in exact order of preference. A matching consists of n distinct pairings of participants that are not ordered. In an instance of SR, a matching M is stable if and only if no two participants x and y prefer each other over their partner in M. A blocking pair is defined as a pair that prevents access to M. When Gale and Shapley first looked at SR, they provided an example that doesn't allow for a stable matching. Irving found a solution to an issue raised by Knuth in his description of an O(n 2) method (henceforth method SR) that finds a stable matching, if one exists, for a given instance of SR. This algorithm is linear in the input size. After that, Gus field and

Irving dug into the structural parts of SR, and they laid out the issue in detail. Moreover, Tan pioneered the idea of a stable partition in SR instance I, which gives a "short certificate" in the event that I denies a stable matching. To identify a stable matching, if one exists, for a given instance of SR, Feder *et al*. recently published an O (n log3 n) parallel approach.
Keep in mind that the standard Stable Marriage issue (SM) has a non-bipartite extension in SR. In SM, the group of people taking part is divided into two separate groups-men and women-and each individual gives the other group members a numerical ranking based on how much they like them. In this case, a same stability definition applies. As far as anybody knows, each SM instance may have a stable matching, and the Gale/Shapley method can find such a matching in linear time.

Incomplete preference lists
To make SR more generalizable, we may let people's preference lists be incomplete (this is called SRI). If participant p is in participant q's preference list, then we say that participant q accepts participant p. Otherwise, we say that participant q rejects participant p. In order for {x, y} to be acceptable, it must be a member of M. This property is necessary for a matching M in an SRI instance. A stable matching M is one in which, given an instance of SRI, no two participants x and y exist such that x is mismatched in M and y is acceptable, or y is preferred to his partner in M. This is the new definition of stability. (We may generalize to say that p is acceptable to q if and only if q is acceptable to p.) This is because, from the perspective of discovering stable matches, there is no need that a stable matching for an SRI instance be a full matching. On the other hand, if one exists, every stable matching for a certain instance is of the same size and matches the same set of participants precisely (Section 4.5.2). Improving Algorithm SR to handle SRI is a piece of cake.

Ties in the preference lists
When people are allowed to indicate ties in their wish lists, another SR generalization follows naturally. Our acronym for "Stable Roommates with Ties" describes this issue. Irving has established three concepts of stability in the setting of SM when ties are permitted (henceforth SMT), and these concepts naturally transfer to SRT. A matching M is considered weakly stable under the weakest of these three conditions if there are no two participants x and y in M 1 who strongly like each other above their partner. As the following thesis shows, the weak stability definition is the same as a tie-breaking condition (the proof of which is unnecessary as it is simple).

Proposition 2: Allow Me to serve as a match in SRT instance I. Then, for each instance of SR acquired from I by breaking the ties, M must be stable in order for M to be weakly stable in I.
In general, there could be an exponential number of methods to break the ties, and various approaches might produce SR instances that allow stable matchings or not. Ronn proved that determining whether a certain SRT case allows a weakly stable matching is an NP-complete task. This work proposes a different, shorter reduction based on a graph matching issue; the proof uses a clever but time-consuming translation from 3SAT.
If there are no two participants x and y, where each person strongly favors or is agnostic about their partner in M, then M is super-stable according to the strongest stability criterion described above. The stability of a very stable matching is obviously low. Another equivalent of Proposition 2 that arises from the super-stability criteria (the proof is again simple and not included) is this one.

Proposition 3: For each instance I of SRT, let M be a matching. If M remains stable in each and every instance of SR produced from I by breaking the ties, then M is super-stable in I.
Even in an SMT case, a super-stable matching is not required (see for more information). Yet, for any given instance of SMT, Irving has developed a linear-time method (henceforth method SMT-Super) to discover super-stable matchings, if any exist. This approach has been recently updated by Irving *et al*. to handle the so-called Hospitals/Residents issue with ties, which is a many-one stable matching problem. It has been up to know to determine if there is an efficient method for SRT under super-stability. Here we provide method SRT-Super, a linear-time method that may be used to check for the existence of a super-stable matching and, if found, to locate it, given an instance of SRT. For an SRT instance without ties, Algorithm SRT-Super reduces to Algorithm SR, and for SMT under super-stability, it is a special case of SRT under superstability, so Algorithm SMT-Super is a subset of Algorithm SR.

The egalitarian stable roommates' problem
We begin by defining the following problems.
Problem 1 EGAL d-SRI
Input: A solvable instance I = G, O of d-SRI, where G is a graph and O is a set of preference lists, each of length at most d.
Output: An egalitarian stable matching M in I.
The decision version of EGAL d-SRI is defined as follows:
Problem 2 EGAL d-SRI DEC
Input: $\mathcal{I} = \langle G, \mathcal{O}, K' \rangle$, where $\langle G, \mathcal{O} \rangle$ is a solvable instance I' of d-SRI and K'
Question: Does I' admit a stable matching M with c(M) ≤ K'? is an integer
In the following we give a reduction from the NP-complete decision version of Minimum Vertex Cover in cubic graphs to EGAL 3-SRI DEC, deriving the hardness of the latter problem.
Theorem 1 EGAL 3-SRI DEC is NP-complete.
Proof Clearly EGAL 3-SRI DEC belongs to NP. To show NP-hardness, we begin by defining the NP-complete problem that we will reduce to EGAL 3-SRI DEC.
Problem 3 3-VC
Input: $\mathcal{I} = \langle G, K \rangle$, where G is a cubic graph and K is an integer.
Question: Does G contain a vertex cover of size at most K?
3-VC is NP-complete
Construction of the EGAL 3-SRI DEC Instance Let $\langle G, K \rangle$ be an instance of 3-VC, where $G = (V, E)$, $E = \{e_1, \ldots, e_m\}$ and $V = \{v_1, \ldots, v_n\}$.

For each $i\ (1 \leq i \leq n)$,

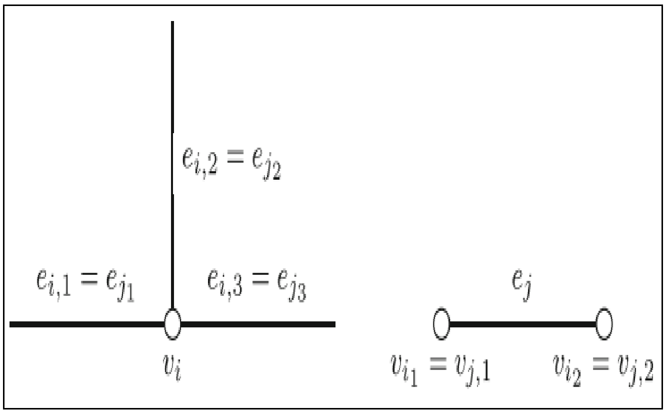

Fig 4: Notation derived from the 3-VC instance $\langle G, K \rangle$

suppose that vi is incident to edges e_{j_1}, e_{j_2} and e_{j_3} in G, were without loss of generality $j_1 < j_2 < j_3$. Define $e_{i,s} = e_{j_s}\ (1 \leq s \leq 3)$. Similarly for each j $(1 \leq j \leq m)$, suppose that $e_j = v_{i_1} v_{i_2}$, where without loss of generality $i_1 < i_2$. Define $v_{j,r} = v_{i_r}\ (1 \leq r \leq 2)$.'

We now construct an instance I of 3-SRI as follows. We define the following sets of vertices.

$$
\begin{aligned}
V' &= \{v_i^r &: 1 \leq i \leq n &\ \wedge\ 1 \leq r \leq 4\} \\
E' &= \{e_j^s &: 1 \leq j \leq m &\ \wedge\ 1 \leq s \leq 2\} \\
W &= \{w_i^r &: 1 \leq i \leq n &\ \wedge\ 1 \leq r \leq 4\} \\
Z &= \{z_i^r &: 1 \leq i \leq n &\ \wedge\ 1 \leq r \leq 4\}
\end{aligned}
$$

Intuitively, $v_i^r \in V'$ corresponds to vertex vi and its incident edge $e_{i,r}$, whilst $e_j^s \in E'$ corresponds to edge ej and its incident vertex vj,s. The set $V' \cup E' \cup W \cup Z$ constitutes the set of agents in I, and the preference lists of the agents are as shown in Fig. 5. In the preference list of an agent vr i (1 ≤ i ≤ n and 1 ≤ r ≤ 3), the symbol

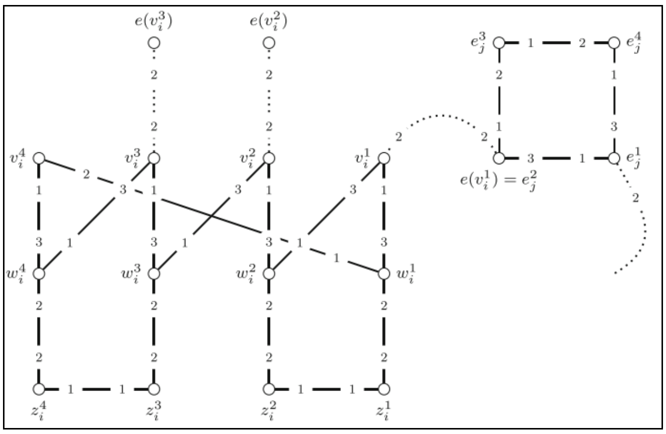

Fig 5: Part of the constructed instance of EGAL 3-SRI DEC

Conclusion

Sotomayor (1996) [12] presented a marriage market version of the idea of simple matching, which is an individually rational matching where the woman in a blocking couple is always unmarried. There is given a proof that does not build anything about the set of stable matchings being non-empty. Showing that the men's Pareto-optimal simple matching is stable is the essence of the evidence. Swapping out males for women and men for women yields the same outcome.

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