



A generalized result on contraction mapping in dislocated quasi-metric spaces

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Abstract

In this paper, we suggest a fixed-point theorem in dislocated quasi metric space for a new continuous contractive mapping. This theorem provides generalization of some existing results in the literature. Our result is supported by example.

Keywords: Metric space, complete metric space, contraction mapping and fixed point

Introduction

The applicability of the fixed point theory has been proved by many authors in several branches of mathematics like optimization theory, game theory, control theory, differential equations, boundary value problems. The study of fixed point theory was started more after Banach fixed point theorem for contraction mapping in complete metric space. This theorem guarantees the existence and uniqueness of fixed point through the application to the certain self-maps of metric spaces. Through the proof theorem, it also provides a constructive method to find the fixed point of self mapping in different versions of generalized metric spaces.

After the Stephan Banach, the fixed point theory on metric spaces has developed through numerous generalizations obtained by strengthening or by weakening the hypothesis in definition of metric spaces. These generalizations of Banach fixed point theorem appear in the theory through the intense interest of different authors to generalize the metric space. It has provided many generalizations by many ways, namely quasi-metric space, dislocated-metric space, partial metric space, dislocated quasi-metric space and generalized quasi-metric spaces. By using concept of partial metric space in 2000, Hitzler and Seda ^[2] worked on metric spaces and excluded the condition of self distance equal to zero in metric spaces to introduce the concept of dislocated metric spaces, where the self distance need not be zero for any point. Further, Hitzler and Seda ^[2] gave the celebrated Banach contraction principle in complete dislocated metric space. The dislocated metric space has proved its importance in logic programming semantics, computer science and electronic engineering.

The notion of quasi-metric, as a generalization of metric spaces, was given by Wilson ^[1] by dropping the symmetric property in dislocated metric space, thereafter Zeyada *et al.* ^[3], introduced some definitions to strengthen the literature of generalization of metric spaces. In their study they used the idea of dislocated metric space due to Hitzler and Seda ^[2] to establish the concept of complete dislocated quasi metric spaces and proved new version of Banach contraction principle fixed point theorem in dislocated quasi-metric spaces. Aage and Salunke ^[4, 5] derived some new fixed point theorems in dislocated and dislocated quasi-metric spaces. Isufati ^[7] derived some fixed point results for continuous contractive conditions in dislocated quasi metric space. In this line, Kohli, *et al.* ^[8], has discussed the existence and uniqueness of a fixed point in a dislocated quasi-metric space.

In the present paper we prove a fixed point result in dislocated quasi-metric spaces for a continuous selfmapping with new contractive conditions our result provides generalization some present results in the literature of dislocated quasi metric spaces.

Preliminaries

Following fundamental definitions, theorems and properties are employed to establish our main result.

Definition 3.1. [3]: Let X be a non-empty set and $d : X \times X \rightarrow [0, \infty)$ be a function satisfying the following conditions:

1. $d(s, s) = 0$,
2. $d(s, t) = d(t, s) = 0$ implies $s = t$,
3. $d(s, t) = d(s, t)$ for all $s, t \in X$,
4. $d(s, t) \leq d(s, u) + d(u, t)$ for all $s, t, u \in X$.

Then d is called a metric on X and (X, d) is called as metric space.

Definition 3.2 [1]: The d satisfying the conditions (i), (ii), (iv) is called as quasi metric on X and (X, d) is known as quasi metric space.

Definition 3.3 [3]: If distance function d satisfies conditions (ii), (iii) and (iv) then d is called as dislocated metric (d -metric) on X .

Definition 3.4 [3]: If d satisfies conditions (ii) and (iv) only then it is called as a dislocated quasi metric (dq -metric) on X .

Remark 3.5 [3] Every metric space on X is a dislocated Metric space on X but not conversely.

Example 3.6 Let $X = \mathbb{R}^+$ and we define the distance function $d : X \times X \rightarrow [0, \infty)$ by $d(s, t) = s + t$ then d is clearly a dislocated metric space but not a metric space.

Remark 3.7 [3] Every dislocated Metric X is a dislocated quasi Metric on X but not conversely.

Example 3.8 Let $X = \mathbb{R}^+$ and we define the distance function $d : X \times X \rightarrow [0, \infty)$ by $d(s, t) = |t|$ then d is clearly a dislocated quasimetric space but not a metric space.

Example 3.9 Let $X = \mathbb{R}^+$ and we define the distance function $d : X \times X \rightarrow [0, \infty)$ by $d(s, t) = \max\{s, t\}$.

Definition 3.10 [3] A sequence $\{s_n\}$ in dq - metric space (X, d) is called Cauchy if for given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$, for all $m, n \geq n_0$

implies $d(s_m, s_n) < \epsilon$ or $d(s_n, s_m) < \epsilon$, $\min\{d(s_m, s_n), d(s_n, s_m)\} < \epsilon$.

Definition 3.11 [3] A sequence $\{s_n\}$ in dq -metric space (X, d) is called 'bi' Cauchy if for given $\epsilon > 0$ such that for all $m, n > n_0$, $\max\{d(s_m, s_n), d(s_n, s_m)\} < \epsilon$.

Definition 3.12 [3] A sequence $\{s_n\}$ in dq - metric space is dislocated quasi-converges to s_0 (or dq converges to s_0) if $\lim_{n \rightarrow \infty} d(s_n, s_0) = \lim_{n \rightarrow \infty} d(s_0, s_n) = 0$ and s_0 is called as dq - limit of sequence $\{s_n\}$ in X .

Proposition 3.13 [5] Every convergent sequence in a dq -metric space is 'bi' cauchy.

Lemma 3.14 [5] Every subsequence of dq -convergent sequence to a point s_0 is dq -convergent to s_0 .

Definition 3.15 [3] A dq -metric space (X, d) is called complete if every Cauchy sequence in (X, d) is a dq -convergent.

Definition 3.16 [3] Let (X, d_α) and (Y, d_β) be two dq -metric spaces and let $T : X \rightarrow Y$ be a function, then T is continuous if for each sequence $\{s_n\}$ which is d_α q-convergent to s_0 in X , the sequence $\{T(s_n)\}$ is $d_\beta q$ -convergent to $T(s_0)$ in Y .

Definition 3.16 [3] Let (X, d) be a dq -metric space, A map $T : X \rightarrow X$ is called contraction if there exists $0 \leq \lambda < 1$ such that $d(Ts, Tt) \leq \lambda d(s, t)$, for all $s, t \in X$.

Lemma 3.17 [3] Every subsequence of dq -convergent sequence to a point s_0 is dq -convergent to s_0 .

Remark 3.18: The converse of Lemma 2.1 may not be true in dq - metric spaces.

Lemma 3.19 ^[3] Let (X, d) be a dq -metric space. If $T : X \rightarrow X$ is a contraction function, then $\{T^n(s_0)\}$ is a Cauchy sequence for each $s_0 \in X$.

Lemma 3.20 ^[3] The dq -limits in a dq -metric space are unique.

Theorem 3.21 ^[3] Let (X, d) be a complete dq -metric space and let $T : X \rightarrow X$ be a continuous contraction function then T has a unique fixed point.

Aage and Salunke ^[4] introduced Kannan-type contraction and derived fixed point theorems. They also has provided generalized contraction in the context of dislocated quasimetric spaces the results are stated in the next theorems.

Theorem 3.22 ^[4]: Let (X, d) be a complete dq -metric space and $T : X \rightarrow X$ be a continuous self-mapping satisfying the following condition:

$$d(Ts, Tt) \leq a \cdot \{d(s, Ts) + d(t, Tt)\}, \text{ where } a \geq 0 \text{ with } a < \frac{1}{2} \text{ and}$$

for all $s, t \in X$ then T has a unique fixed point.

Theorem 3.23 ^[4] Let (X, d) be a complete dq -metric space and $T : X \rightarrow X$ be a continuous self-mapping satisfying the following condition:

$$d(Ts, Tt) \leq a \cdot d(s, t) + b \cdot d(s, Ts) + c \cdot d(t, Tt) + e \cdot \{d(s, Tt) + d(t, Ts)\},$$

Where $a, b, c, e \geq 0$ with $a + b + c + 2e < 1$ and for all $s, t \in X$ then T has a unique fixed point.

Further, Aage and Salunke ^[5] established the following results for continuous mappings in dislocated quasi metric spaces.

Theorem 3.24 ^[5] Let (X, d) be a complete dq -metric space and $T : X \rightarrow X$ be a continuous self-mapping satisfying the following condition:

$$d(Ts, Tt) \leq a \cdot d(s, t) + b \cdot d(s, Ts) + c \cdot d(t, Tt),$$

Where $a, b, c \geq 0$ with $a + b + c < 1$ and for all $s, t \in X$ Then T has a unique fixed point.

Isufati ^[7] proved some fixed point theorems for continuous contraction mappings defined by Dass and Gupta [6] in dislocated quasi-metric spaces.

Theorem 3.25 ^[6] Let (X, d) be a complete dq -metric space and $T : X \rightarrow X$ be a continuous self-mapping satisfying the following condition:

$$d(Ts, Tt) \leq a \cdot \frac{d(t, Tt)[1 + a(s, Ts)]}{1 + a(s, t)} + b \cdot d(s, t),$$

Where $a, b > 0$ with $a + b < 1$ and for all $s, t \in X$. Then T has a unique fixed point.

Kohli, *et al.* ^[13] considered dislocated quasi metric spaces and proved fixed point theorem in the space stated below.

Theorem 3.26 ^[13] Let (X, d) be a complete dq -metric space and $T : X \rightarrow X$ be a continuous self-mapping satisfying the following condition:

$$d(Ts, Tt) \leq a \cdot \frac{d(t, Tt)[1 + d(s, Ts)]}{1 + d(s, t)} + b \cdot d(s, t) + c \cdot d(t, Tt),$$

Where $a, b, c > 0$ with $a + b + c < 1$ and for all $s, t \in X$ then T has a unique fixed point.

Motivated from Aage and Salunke ^[4], Aage and Salunke ^[5], Isufati ^[7], Kohli, *et al.* ^[8], we present a continuous contractive mapping and prove fixed point theorem in dislocated quasi metric space and prove existence and uniqueness of fixed point in dislocated quasi metric spaces for this contractive mapping.

Main Result

In this section we prove our main result.

Theorem 4.1: Let (X, d) be a complete dislocated quasi metric space (dq -metric space) and $T : X \rightarrow X$ be a continuous self mapping satisfying the following condition

$$d(Ts, Tt) \leq \alpha_1 d(s, t) + \alpha_2 d(s, Ts) + \alpha_3 d(t, Tt) + \alpha_4 \frac{d(s, Ts)d(t, Tt)}{d(s, t)} + \alpha_5 \frac{d(s, Tt)}{[1+d(s, t)]} + \alpha_6 \frac{d(t, Tt)[1+d(s, Ts)]}{1+d(s, t)}$$

Equation 4.1

With $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6 < 1$ where, for each $\alpha_i \geq 0, 1 \leq i \leq 6$ for all $s, t \in X$ then T has unique fixed point in X .

Proof: Let us choose s_0 an arbitrary member in X and define a sequence $\{s_n\}$ by $s_0, s_1 = T(s_0), s_2 = T(s_1), \dots, s_n = T(s_{n-1})$ for all $n \in N$.

To prove that, $\{s_n\}$ is a Cauchy sequence in X .

Consider,

$$d(s_n, s_{n+1}) = d(Ts_{n-1}, Ts_n) \leq \alpha_1 d(s_{n-1}, s_n) + \alpha_2 d(s_{n-1}, Ts_{n-1}) + \alpha_3 d(s_n, Ts_n) + \alpha_4 \frac{d(s_{n-1}, Ts_{n-1})d(s_n, Ts_n)}{d(s_{n-1}, s_n)} + \alpha_5 \frac{d(s_{n-1}, Ts_n)}{1 + d(s_{n-1}, s_n)} + \alpha_6 \frac{d(s_n, Ts_n)[1 + d(s_{n-1}, Ts_{n-1})]}{1 + d(s_{n-1}, s_n)}$$

$$d(s_n, s_{n+1}) \leq \alpha_1 d(s_{n-1}, s_n) + \alpha_2 d(s_{n-1}, s_n) + \alpha_3 d(s_n, s_{n+1}) + \alpha_4 \frac{d(s_{n-1}, s_n)d(s_n, s_{n+1})}{d(s_{n-1}, s_n)} + \alpha_5 \frac{d(s_{n-1}, s_{n+1})}{1 + d(s_{n-1}, s_n)} + \alpha_6 d(s_n, s_{n+1})$$

$$d(s_n, s_{n+1}) \leq \alpha_1 d(s_{n-1}, s_n) + \alpha_2 d(s_{n-1}, s_n) + \alpha_3 d(s_n, s_{n+1}) + \alpha_4 d(s_n, s_{n+1}) + \alpha_5 \frac{d(s_{n-1}, s_{n+1})}{1 + d(s_{n-1}, s_n)} + \alpha_6 d(s_n, s_{n+1})$$

$$d(s_n, s_{n+1}) \leq \alpha_1 d(s_{n-1}, s_n) + \alpha_2 d(s_{n-1}, s_n) + \alpha_3 d(s_n, s_{n+1}) + \alpha_4 d(s_n, s_{n+1}) + \alpha_5 d(s_{n-1}, s_{n+1}) + \alpha_6 d(s_n, s_{n+1})$$

$$d(s_n, s_{n+1}) \leq \alpha_1 d(s_{n-1}, s_n) + \alpha_2 d(s_{n-1}, s_n) + \alpha_3 d(s_n, s_{n+1}) + \alpha_4 d(s_n, s_{n+1}) + \alpha_5 \{d(s_{n-1}, s_n) + d(s_n, s_{n+1})\} + \alpha_6 d(s_n, s_{n+1})$$

$$(1 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6)d(s_n, s_{n+1}) \leq (\alpha_1 + \alpha_2 + \alpha_5) d(s_{n-1}, s_n)$$

$$d(s_n, s_{n+1}) \leq \frac{(\alpha_1 + \alpha_2 + \alpha_5)}{(1 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6)} d(s_{n-1}, s_n)$$

Let $k = \frac{(\alpha_1 + \alpha_2 + \alpha_5)}{(1 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6)}$

Therefore, $d(s_n, s_{n+1}) \leq kd(s_{n-1}, s_n)$

Similarly, we can have,

$$d(s_{n-1}, s_n) \leq kd(s_{n-2}, s_{n-1})$$

$$d(s_n, s_{n+1}) \leq k^2 d(s_{n-2}, s_{n-1})$$

Thus, $d(s_n, s_{n+1}) \leq k^n d(s_0, s_1)$

Now, for any two natural numbers m, n satisfying $m > n$ and employing the triangle inequality in dq metric space. we can have,

$$\begin{aligned} d(s_n, s_m) &\leq d(s_n, s_{n+1}) + d(s_{n+1}, s_{n+2}) + \dots + d(s_{m-1}, s_m) \\ &\leq k^n d(s_0, s_1) + k^{n+1} d(s_0, s_1) + k^{n+2} d(s_0, s_1) + \dots + k^{m-1} d(s_0, s_1) \\ &\leq (k^n + k^{n+1} + k^{n+2} + \dots + k^{m-1}) d(s_0, s_1) \end{aligned}$$

$$d(s_n, s_m) \leq \left(\frac{k^n}{1-k} \right) d(s_0, s_1)$$

But given condition is that, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6 < 1$

This implies that, $k = \frac{(\alpha_1 + \alpha_2 + \alpha_5)}{(1 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6)} < 1$

Also, for each $\alpha_i \geq 0, 1 \leq i \leq 6$.

and we are having, $0 \leq k < 1$ and $k^n \rightarrow 0$ as $n \rightarrow \infty$.

For each $\epsilon > 0$, we choose $N_0 \geq 0$, in such a way that,

$$\left(\frac{k^n}{1-k} \right) d(s_0, s_1) < \epsilon$$

then for any choice of m and n with $n, m \geq N_0$,

We get that,

$$d(s_n, s_m) \leq \left(\frac{k^n}{1-k} \right) d(s_0, s_1) \leq \left(\frac{k^{N_0}}{1-k} \right) d(s_0, s_1) < \epsilon$$

Also, following the same steps as above we can prove that, $d(s_m, s_n) < \epsilon$

Thus, from $d(s_n, s_m) < \epsilon$ and $d(s_m, s_n) < \epsilon$ it shows that, sequence $\{s_n\}$ is a Cauchy's sequence in complete dq -metric space (X, d) .

Since, (X, d) is dq -complete metric space, the sequence $\{s_n\}$ must converge in X .

Therefore, there must exists a point $\xi \in X$ such that, $s_n \rightarrow \xi$ as $n \rightarrow \infty$.

i.e. ξ is a limit point of $\{s_n\}$

Since, T is continuous, we can have $T\xi = T\left(\lim_{n \rightarrow \infty} s_n\right) = \lim_{n \rightarrow \infty} T(s_n) = \lim_{n \rightarrow \infty} s_{n+1} = \xi$

Giving finally that, $T\xi = \xi$, and this proves that ξ is fixed point of T .

To prove uniqueness of fixed point for T

We assume that, T has two fixed point namely ξ and ψ and then we have $T\xi = \xi$ and $T\psi = \psi$.

From equation 3.1 $d(\xi, \xi) = 0$ and $d(\psi, \psi) = 0$.

Consider,

$$\begin{aligned} d(\xi, \psi) &= d(T\xi, T\psi) \\ &\leq \alpha_1 d(\xi, \psi) + \alpha_2 d(\xi, T\xi) + \alpha_3 d(\psi, T\psi) \\ &+ \alpha_4 \frac{d(\xi, T\psi)d(\psi, T\psi)}{d(\xi, \psi)} + \alpha_5 \frac{d(\xi, T\psi)}{[1 + d(\xi, \psi)]} + \alpha_6 \frac{d(\psi, T\psi)[1 + d(\xi, T\xi)]}{1 + d(\xi, \psi)} \end{aligned}$$

$$d(\xi, \psi) \leq \alpha_1 d(\xi, \psi) + \alpha_2 d(\xi, \xi) + \alpha_3 d(\psi, \psi) + \alpha_4 \frac{d(\xi, \psi)d(\psi, \psi)}{d(\xi, \psi)} + \alpha_5 \frac{d(\xi, \psi)}{[1 + d(\xi, \psi)]} + \alpha_6 \frac{d(\psi, \psi)[1 + d(\xi, \xi)]}{1 + d(\xi, \psi)}$$

$$d(\xi, \psi) \leq \alpha_1 d(\xi, \psi) + \alpha_2 d(\xi, \xi) + \alpha_3 d(\psi, \psi) + \alpha_4 d(\psi, \psi) + \alpha_5 d(\xi, \psi) + \alpha_6 \frac{d(\psi, \psi)[1 + d(\xi, \xi)]}{1 + d(\xi, \psi)}$$

$$d(\xi, \psi) = d(T\xi, T\psi) \leq \alpha_1 d(\xi, \psi) + \alpha_5 d(\xi, \psi)$$

$$d(\xi, \psi) \leq (\alpha_1 + \alpha_5) d(\xi, \psi)$$

This leads to $d(\xi, \psi) \leq 0$, Since, $0 \leq (\alpha_1 + \alpha_5) < 1$

But $d(\xi, \psi) \geq 0$ proving that, $d(\xi, \psi) = 0$

Equation 4.2

and similarly we can prove that, $d(\psi, \xi) = 0$

Equation 4.3

Thus, from Equation 4.2 and Equation 4.3 we get, $\xi = \psi$.

This proves uniqueness of fixed point.

Example 4.2 Let us consider $X = [0,1]$ a complete dislocated quasi metric space defined by $d(s, t) = |s|$ for all, $s, t \in X$ and T continuous self-mapping defined

$$\text{As } Ts = \frac{s}{4} \text{ With } \alpha_1 = \frac{1}{6}, \alpha_2 = \frac{1}{8}, \alpha_3 = \frac{1}{12}, \alpha_4 = \frac{1}{14}, \alpha_5 = \frac{1}{16}, \alpha_6 = \frac{1}{18}$$

Then T is self-mapping satisfying the conditions mentioned in Theorem 4.1 and $s = 0$ is the unique fixed point of T in X .

Remark 4.3

In theorem 4.1

1. If we set $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$ then we get Theorem 2.1 in Zeyada *et al.* [3].
2. If we set $\alpha_1 = \alpha_4 = \alpha_5 = \alpha_6 = 0$ with $\alpha_2 = \alpha_3 = \alpha$ and $\alpha < \frac{1}{2}$ and then we get Theorem 3.3 in Aage and Salunke [4].
3. If we set $\alpha_4 = \alpha_5 = \alpha_6 = 0$ then we get Theorem 3.3 in Aage and Salunke [5].
4. If we set $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$ then we get Theorem 3.1 in Isufati [7].
5. If we set $\alpha_2 = \alpha_3 = \alpha_4 = 0$ then we get Theorem 1 in Kohli *et. al.* [8].

Conclusion

The fixed point theorem, Theorem 4.1 proved in this paper provides an extension of Zeyada *et al.* [3], Aage and Salunke [4], Aage and Salunke [5], Isufati [7], Kohli *et al.* [8].

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