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Practical Applications of Group Actions in Physics, Chemistry, And Coding Theory

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Abstract

Group actions serve as a bridge between group theory and geometry, enabling the study of symmetries and transformations in various mathematical structures. In abstract algebra and other branches of mathematics, the concept of a large group of individuals collaborating on a gives a strong foundation for comprehending how a group can symmetrically transform the elements of a set. The thesis centers on exploring and resolving the complexities associated with group actions on sets, particularly in terms of their advanced theoretical aspects and practical applications. The aim is to deepen the understanding of how group actions operate in more complex scenarios, develop new applications in various mathematical and scientific fields, and address the challenges in computational group theory and interdisciplinary applications.

Keywords: Practical Applications, Group Actions in Physics, Chemistry, and Coding Theory

Introduction

In group theory and combinatorics, Burnside's Lemma-sometimes called the Cauchy-Frobenius Theorem-is a basic conclusion. Taking into account the number of items left unmodified (fixed) by each group element, it provides an approach to quantifying the quantity of distinct items subject to a collective operation. This theorem is particularly useful in cases where direct counting is difficult due to symmetrical redundancy. The key insight of Burnside's Lemma is that instead of counting objects directly, we count how many times each object is fixed under different symmetries and average the results. This eliminates redundant counting and provides an accurate count of unique configurations.

The idea of group actions is basic and has many uses in many branches of mathematics and beyond

- **Symmetry in Geometry:** Group actions are used to study symmetries of geometric objects. For example, when a regular polygon's set of rotations is applied to its vertices, revealing its symmetrical structure.
- **Algebraic Structures:** In algebra, group actions help to understand the structure of algebraic objects, including rings, fields, and vector spaces. The generic linear group's action on a vector space, for instance, is useful to analyze linear transformations.
- **Permutation Groups:** Group actions provide a framework for studying permutation groups, which are groups that describe the rearrangements of a set. This is crucial in combinatorics and the study of symmetric groups.
- **Galois Theory:** Group actions are central to Galois theory, when a Galois group's action on a polynomial's roots aids in solve equations and understand the structure of fields.
- **Physics and Chemistry:** In physics and chemistry, group actions describe the symmetries of physical systems, such as molecules and crystals, leading to insights into their properties and behaviors.
- **Coding Theory:** Group coding theory involves activities, which are used to build error-correcting codes by exploiting symmetries in the code structure.

Group actions on sets provide a versatile and powerful tool for analyzing the symmetries and transformations of various mathematical and physical structures. By bridging abstract group theory with concrete applications, group actions facilitate a deeper understanding of the underlying principles governing diverse systems in mathematics, science, and engineering. As a result, the study of group actions continues to be a rich and fruitful area of research with far-reaching implications.

Applications in Coding Theory and Error Correction

Group actions help classify and construct error-correcting codes used in data transmission. In coding theory, Burnside's Lemma helps count unique codewords when symmetries (such as cyclic shifts) are considered equivalent. For example, linear block codes use group actions to classify equivalent codes that can be transformed into one another by permutations of symbols. This helps in optimizing storage and transmission efficiency in digital communication.

Symmetric group actions provide powerful tools for solving combinatorial counting problems. Burnside's Lemma and related results allow us to count objects efficiently when symmetries must be considered. These techniques have broad applications in graph theory, necklace counting, molecular chemistry, coding theory, and computer science, making them essential in both pure and applied mathematics.

Literature Review

K. N. Raghavan (2012) [1] At its core, the idea of a group operating on a set, if not more so, than the idea of a group itself, as groups are only interesting when they do something. This concept is defined and shown first. Orbits and stabilisers help us understand the structure of an activity. We provide an overview of them and provide some thoughts on them. These insights are so basic that they should be obvious on their own, yet when applied correctly, they have intriguing results. The first technical conclusion of these notes is the class equation, which follows a discussion of Lagrange's theorem in terms of group actions. We utilize the class equation to obtain several well-known fundamental findings example: the fact that the centre of a p-group is not trivial in finite group theory, the subgroup existence theorems of prime power orders, and the results of Sylow's theorems regarding the number and conjugacy of subgroups bearing his name.

Ryan C. Spieler (2017) [2] Group activities are the focus of this article. The most elementary algebraic entities are groups, which are sets that can only undergo one operation. To start, we'll take a closer look at their definitions and examine some important ones that pertain to groups. After that, we'll go over several key ideas associated with group activities and provide definitions for them. The rest of the study delves into two significant action kinds and utilizes them to investigate finite group structure. The coset comes first. We shall demonstrate Lagrange's Theorem, a significant outcome in Finite Group Theory, by investigating cosets. Conjugation is the second action type that will be discussed. By using what we know about conjugation, we shall demonstrate Sylow's Theorems, a collection of claims that reveal the inner workings of a finite group.

Eric J. Pap (2024) [3] We discuss fiber bundles in which the fibers are a free G-space with discontinuous orbits rather than a group G. We prefer to refer to these fibers as semi-torsors since they are not torsors per se but rather the result of combined unions of many of them. The term "semi-principal bundle" describes a natural generalization of principal bundles to bundles of semi-torsors. Similar to how primary bundles allow parallel transit, these bundles do as well. The key distinction is that lifts have the potential to land in a different group orbit, rendering group translations insufficient to characterize the transformation. By establishing the concept of a G-set basis, similar to a vector space basis, such effects may be studied more easily. Parallel transport entails rearranging and scaling the basic elements with the relevant group elements because these are the orbits' reference points. A wreath product group describes these two basis symmetries. The traditional approach is made possible by the primary frame bundle that follows from the idea of basis. Furthermore, A retraction from the semi-principal bundles to the principal bundles is shown to be the frame bundle functor. For example. Geometric phases and exceptional points in adiabatic quantum mechanics may be unified mathematically by using the theory that is provided.

Dongxian Tang (2022) [4] Abstract Algebra relies heavily on groups; in fact, many algebraic structures, such as rings, fields, and modules, may be considered as constructed by introducing additional axioms and operations based on groups. Group theory is a common tool for researchers looking to explain a wide range of phenomena. Crystallography has recently made use of group theory to delve further into the mathematical aspects of macroscopic crystal symmetry. This article will explore the uses of group theory in crystallography and magic cubic. We show the most fundamental models and definitions of these areas. The idea of finite groups is fundamental to group theory, here defined as sets with a maximum allowable size. In addition, the n-1 elements of an n-order group may determine the n/o element entirely, and the article provides a mechanism for the n/o element in an n-order commutative group. Group theory seems to have a significant impact on other fields, according to the data.

Orgest Zaka (2011) [5] We shall present a few group theory applications in this presentation. The first use case takes use of the fact that, for high enough n, computing in Zn may completely replace computing in Z. We can do the calculations in parallel in the smaller components of Zn to the extent that it is reducible to prime power order groupings. Group theory's fascinating application to software design is the development of computing algorithms that significantly accelerate computations; this article provides one such example. Symmetries, revolutions, and many geometric transactions may be studied with the help of group theory, which will be introduced in this article. We will also demonstrate some interesting group theory applications in chemistry.

Hopf-Galois Structures: Z Groups of the type $\mathbb{Z}_n \rtimes \mathbb{Z}_2$ are fully described in this part, along with some fundamental number theoretic findings in order to compile a list of the standard embeddings for use in part 3.

Forming groups $\mathbb{Z}_n \rtimes_\phi \mathbb{Z}_2, n$ odd

Observe that in the event that $n = \prod_{t=1}^{m} p_t^{\alpha_t}$; where p_i 's are all distinct primes, then

$$\mathbb{Z}_n \cong \bigoplus_{t=1}^{m} \mathbb{Z}_{p_t^{\alpha_t}},$$

$$\text{and } \operatorname{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^* \cong \prod_{t=1}^{m} \mathbb{Z}_{p_t^{\alpha_t}}^*$$

$$\cong \bigoplus_{t=1}^{m} \mathbb{Z}_{p_t^{\alpha_t - 1}(p_t - 1)}.$$

For $x \in \mathbb{Z}_n$ we have $x = (x_1, x_2, \cdots, x_m)$ where $x_u \in \mathbb{Z}_{p_u^{\alpha_u}}$. We define $p_u(x) = x_u$ for $p_u \in \pi(n)$. If $\phi : \mathbb{Z}_2 = \{\pm 1\} \to Aut(\mathbb{Z}_2)$ is homomorphic to a group that contains $p_u(\phi(-1)(x)) = p_u x$ on behalf of each $p_u \in \pi(n)$, then $\mathbb{Z}_n \rtimes \phi\, \mathbb{Z}_2$ as the 2n-dimensional dihedral group, here abbreviated as D. At the time $p_u(\mathit{The\ integral\ of}\ \phi(-1)(x)) = p_u(x)$ for all $p_u \in \pi(n)$, then $\mathbb{Z}_n \rtimes \phi\, \mathbb{Z}_2$ may be represented by the letter C, and it is the 2n-th order cyclic group. Just pretend for a second that $p_u(\phi(-1)(x)) = p_u(x)$ for a few $p_u \in \pi(n)$ and $p_u'(\phi(-1)(x)) = -p_u'(x)$ for a few $p_u' \in$ The set is isomorphic to π(n) because of this $D_{2k} \times Cl$ for some $k, l \in N$ with $kl = n$.

Basic results

Lemma 1: Since γ because it divides p, we may state that p is a prime number greater than 2. Define $f_\gamma(0) = 0$ Because with every $\delta \in \mathbb{Z} > 0$ define

$$f_\gamma(\delta) = \sum_{i=0}^{\delta-1} \gamma^i.$$

Then,

$$f_\gamma(\delta_1) \equiv f_\gamma(\delta_2) \mod p^n \ \mathit{iff}\ \delta_1 \equiv \delta_2 \mod p^n.$$

Proof: Behold the demonstration of Lemma.

Corollary 2: For any set Z, there exists a prime integer p and a non-zero-member b such that $b^{p^m} \equiv 1 \bmod p^n$. Then

$$p^m | f_b(p^m)\, and\ p^{m+1} \nmid f_b(p^m).$$

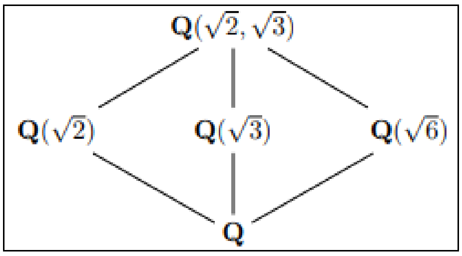
Proof. This is derived from the finding that $b \equiv 1 \bmod p$.

Applications of Galois Theory

Example 1: Its Galois group has order 4 because q(2,3)/q is a Galois extension of degree 4. The values of the components of the Galois group on $\mathbb{Q}$ √ 2 and 3 are used to determine them. Since ± √ 2 and ± √ 3 are the Q-conjugates of √ 2 and √ 3, respectively, the Galois group can only accommodate a maximum of four potential automorphisms. Check out the first table. Since All four members of the Galois group are of order 4. of these potential values for σ(√2) and σ(√3) are valid.

σ	$\sigma(\sqrt{2})$	$\sigma(\sqrt{3})$
id.	$\sqrt{2}$	$\sqrt{3}$
σ_1	$\sqrt{2}$	$-\sqrt{3}$
σ_2	$-\sqrt{2}$	$\sqrt{3}$
σ_3	$-\sqrt{2}$	$-\sqrt{3}$

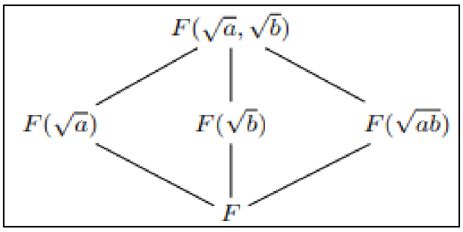
The automorphisms σ1, σ2, and σ3 purchase two. The product of Q(√ 2, √ 3) and three subfields K_i gives a product of 2, or put another way, [K_i: Q] = 4/2 = 2, since the product of Gal(Q(√ 2, √ 3)/Q) and Q(√ 2, √ 3) contains three elements of order 2. Q(√2) and Q(√3) are two examples of such fields. Q(√6) rounds out the list, and a third is also present. All the subfields are shown in this figure.



The subgroup fixing Given that (− √ 2) (− √ 3) = √ 2 √ 3, the subgroup fixing Q (√ 6) is {σ_3}, the subgroup fixing Q(√ 2) is hσ1i, and the subgroup fixing Q(√ 3) is {σ_2}. Table 2 shows the impact about the Gal (Q (√ 2, √ 3)/Q) on √ 2 + √ 3. The four numbers are separate from one another because squares 2 and 3 are linearly independent over Q. For this reason, Q(√ 2, √ 3) is the same as Q(√ 2 + √ 3). ($X → (√ 2 + √ 3)) (X in the direction of (-1/2) plus (3/2)) The equation (X − (− 2 − √ 3)) equals (minimal polynomial of √ 2 + √ 3 over Q) X^4 − 10X^2 + 1.

σ	$\sigma(\sqrt{2})$	$\sigma(\sqrt{3})$	$\sigma(\sqrt{2}+\sqrt{3})$
id.	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{2}+\sqrt{3}$
σ_1	$\sqrt{2}$	$-\sqrt{3}$	$\sqrt{2}-\sqrt{3}$
σ_2	$-\sqrt{2}$	$\sqrt{3}$	$-\sqrt{2}+\sqrt{3}$
σ_3	$-\sqrt{2}$	$-\sqrt{3}$	$-\sqrt{2}-\sqrt{3}$

Specifically, as a minimum polynomial over Q, $X^4 - 10X^2 + 1$ cannot be reduced to any element of Q[X]. Same line of thinking applies if a field F is not characteristic 2 and a and b are This picture illustrates that all the fields between F and F (√ a, √ b) are the same when there are no squares in F such that ab is also not a square in F. As a consequence, [F (√ a, √ b): F] is the outcome. = 4.

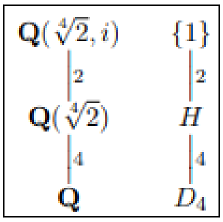


In addition, it is equivalent to the function F (√ a + √ b) that the function F ($\sqrt{a}$, $\sqrt{b}$) takes on. Similar to the preceding specific case, the reasoning.

Example 2: While Q ($\sqrt[4]{2}$)/Q lacks Boolean properties, Q($\sqrt[4]{2}$) appears in Q ($\sqrt[4]{2}$, i), All the way to Q, this is Galois. We shall use Galois theory to: Q($\sqrt[4]{2}$, i), /Q such that we may find the gaps between Q($\sqrt[4]{2}$). A collection of Galois operators on Q($\sqrt[4]{2}$, i)/Q is equivalent to $\{r, s\}$ in which

$$r(\sqrt[4]{2}) = i\sqrt[4]{2},\ r(i) = i \text{ and } s(\sqrt[4]{2}) = \sqrt[4]{2},\ s(i) = -i.$$

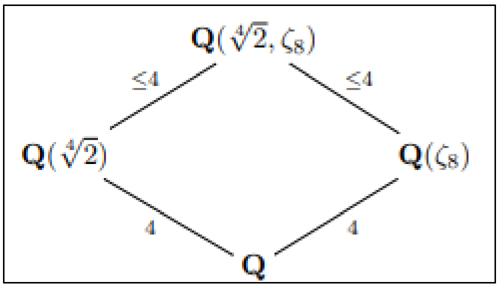
(While keeping an eye on Q($\sqrt[4]{2}$, i) S treats complex numbers similarly to complex conjugation since they are both number systems. We find D_4, an isomorphism, in the set {r,s}. revolves around a square at a 90-degree angle and s is a reflection across a diagonal. Can you tell me Gal's subgroup H? (Q($\sqrt[4]{2}$, i)/Q) that are similar to Q($\sqrt[4]{2}$)?



Given that s is a nontrivial member of the Galois group who fixes Q($\sqrt[4]{2}$), s ∈ H. How big is H? [Q Q($\sqrt[4]{2}$), i): Q($\sqrt[4]{2}$)= 2, so $H = \{1, s\} = \{s\}$. Using the Galois correspondence to find Q($\sqrt[4]{2}$, i)/Q, areas that fall squarely inside Q($\sqrt[4]{2}$) on the interval between hsi and hr, si, and Q belonging to the Galois group. Based on the well-established subgrouping structure of D_4, the only subset that is entirely inside {s} and {r,s} is {r^2,s} Therefore, there is exactly one area that is classified as falling between Q($\sqrt[4]{2}$), and Q. This is the only one since Q (2) is a field of this kind.

Remark 3: Although there may be other arguments, Galois theory does provide the most methodical approach to discovering intermediate domains. To illustrate, let's say Q ⊂ F ⊂ Q($\sqrt[4]{2}$) by setting [F: Q] = 2. After that $\sqrt[4]{2}$ is 2. °F more than F because $\sqrt[4]{2}$ originates from $X^4 - 2$, A quadratic factor of must be a minimum polynomial over F $X^4 - 2$. Three distinct quadratic elements are present, and they are $\sqrt[4]{2}$ in the base, but only one of, $X^2 - \sqrt{2}$, contains values for the in Q($\sqrt[4]{2}$) to say nothing of R. Therefore $X^2 - \sqrt{2}$ should represent the minimum polynomial of $\sqrt[4]{2}$ over F, so $\sqrt{2}$ ∈ F. Since [F: Q] = 2, F = Q ($\sqrt{2}$) while keeping track of degrees.

Example 4: Consider Q($\sqrt[4]{2}$, ζ_8), where $\zeta_8 = e^{2\pi i/8}$ the smallest polynomial over Q of an 8-th root of unity is $X^4 + 1$. Both Q($\sqrt[4]{2}$) and Q(ζ_8) are four degrees above Q. Given that ζ_8^2 = i, Q($\sqrt[4]{2}$, ζ_8), is an area that segmented Q into ($X^4 - 2$) ($X^4 + 1$) hence has an extension over Q that is Galois. May I ask what this Galois group is? A field diagram that follows is ours.



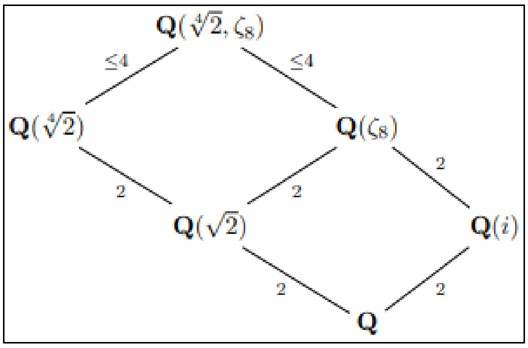
Thus [Q($\sqrt[4]{2}$, ζ_8): Q] exceeds 16. It turns that the degree isn't 16 after all, thanks to some obscure algebraic relationships when considering both $\sqrt[4]{2}$, and ζ_8. Any σ ∈ Gal (Q($\sqrt[4]{2}$, ζ_8))/Q) includes values that define it.

$$\sigma(\zeta_8) = \zeta_8^a \quad (a \in (\mathbf{Z}/8\mathbf{Z})^\times) \quad \text{and} \quad \sigma(\sqrt[4]{2}) = i^b\sqrt[4]{2} \quad (b \in \mathbf{Z}/4\mathbf{Z}).$$

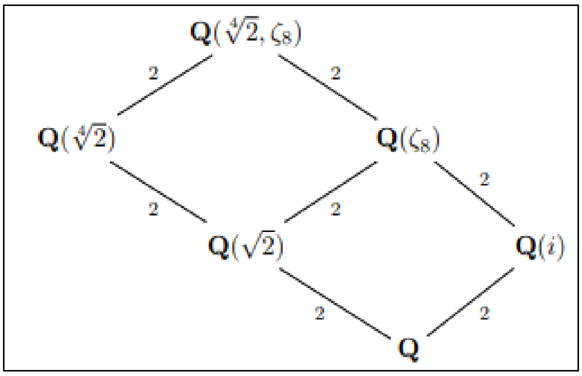
For both a and b, four options are available. Assuming free choices of a and b, the Galois group contains no more than 16 automorphisms. On the other hand, you can't choose options a and b separately because ζ_8 and $\sqrt[4]{2}$ are interconnected:

$$\zeta_8 + \zeta_8^{-1} = e^{2\pi i/8} + e^{-2\pi i/8} = 2\cos\left(\frac{\pi}{4}\right) = \sqrt{2} = \sqrt[4]{2}^2.$$

This indicates that both parties own $\sqrt{2}$ Q (ζ_8) and Q($\sqrt[4]{2}$). The common subfield Q ($\sqrt{2}$) is shown in this field diagram Q($\sqrt[4]{2}$) and Q (ζ_8).



Rewriting $\zeta_8 + \zeta_8^{-1} = \sqrt{2}$ as $\zeta_8^2 - \sqrt{2}\zeta_8 + 1 = 0$, ζ_8 earned no more than two degrees in Q ($\sqrt[4]{2}$). Since ζ_8 doesn't exist; it has no interior Q($\sqrt[4]{2}$), hence Degree 2 is more than Q($\sqrt[4]{2}$). Therefore [Q($\sqrt[4]{2}$, ζ_8): Q] = 2 · 4 = 8 2 represents the degree value that is shown as "⪇ 4" in the diagram. A new field diagram is shown below.



Revisiting the Galois group, the impact of is given by (1.3). $\sigma \in \mathrm{Gal}(\mathbf{Q}(\sqrt[4]{2},\zeta_8)/\mathbf{Q})$ on $\sqrt[4]{2}$ portion of what decides it on ζ_8, in contrast to: $(\sigma(\sqrt[4]{2}))^2 = \sigma(\zeta_8)+\sigma(\zeta_8)^{-1}$, which in the notation of (1.2) is the same as

$$(-1)^b = \frac{\zeta_8^a+\zeta_8^{-a}}{\sqrt{2}}.$$

Since $\zeta_8 = e^{2\pi i/8} = (1+i)/\sqrt{2}$, a calculation shows $\zeta_8^a+\zeta_8^{-a} = \sqrt{2}$ if $a \equiv 1, 7$ mod 8 and $\zeta_8^a+\zeta_8^{-a} = -\sqrt{2}$ if a ≡ 3, 5 mod 8 (note $\zeta_8^7=\zeta_8^{-1}$ and $\zeta_8^5=\zeta_8^{-3}$). If an is less than or equal to 1, then, 7 mod 8 gives us $(-1)^b$ Because $(-1)^b = -1$ in 5 mod 8, if a ≡ 3, then b ≡ 1, 3 mod 4. On the other hand, if a = 1, then b ≡ 0, 2 mod 4. As an example, σ is unable to rectify $\sqrt[4]{2}$ (b = 0) while transmitting ζ_8 to ζ_8^3 (a = 3) so (1) would be invalid

One of the easiest ways to comprehend Q($\sqrt[4]{2}$, ζ_8) involves switching to another set of power plants. Considering as

$$\zeta_8 = e^{2\pi i/8} = e^{\pi i/4} = (1+i)/\sqrt{2},$$

$$\mathbf{Q}(\sqrt[4]{2},\zeta_8) = \mathbf{Q}(\sqrt[4]{2}, i),$$

Since $\zeta_8 = e^{2\pi i/8} = (1+i)/\sqrt{2}$, a calculation shows $\zeta_8^a+\zeta_8^{-a} = \sqrt{2}$ if a ≡ 1, 7 mod 8 and $\zeta_8^a+\zeta_8^{-a} = -\sqrt{2}$ if a ≡ 3, 5 mod 8 (note $\zeta_8^7=\zeta_8^{-1}$ and $\zeta_8^5=\zeta_8^{-3}$). Therefore, if an is equal to 1, 7, mod 8, then b is equal to 0, 2 mod 4, and vice versa if an is equal to 3, 5 mod 8, then b is equal to 1, 3 mod 4. To illustrate, σ is unable to rectify $\sqrt[4]{2}$ (b = 0) while transmitting ζ_8 to ζ_8^3 (a = 3) for the reason why (1.4) is not valid. One of the easiest ways to comprehend Q($\sqrt[4]{2}$, ζ_8) involves switching to a new set of generators. Given that $\zeta_8 = e^{2\pi i/8} = e^{\pi i/4} = (1+i)/\sqrt{2}$,

$$\mathbf{Q}(\sqrt[4]{2},\zeta_8) = \mathbf{Q}(\sqrt[4]{2}, i),$$

Additionally, we may determine that the second representation's Isomorphism between Galois groups over Q and D_4 while retaining autonomy over the distribution $\sqrt[4]{2}$ i (to any square root of -1), instead of $\sqrt[4]{2}$ and ζ_8. The appearance of the Galois group may be clarified by switching field generators. Furthermore, it is instantly apparent as seen in the second example, [Q($\sqrt[4]{2}$, ζ_8): Q] = 8. Our actions will be based on the preceding example, where [Q($\sqrt[4]{2}$, ζ_8): Q] gives a result of 8, not 16, when passed via [Q($\sqrt[4]{2}$): Q] and [Q(ζ_8): Q]), supposing that Q($\sqrt[4]{2}$) and Q(ζ_8) contain √2 and are hence not "independent" on Q. Given that ζ_8 6∈ Q($\sqrt[8]{2}$) but $\zeta_8^2 - \sqrt{2}\zeta_8 + 1 = 0$ (1.3) the algebraic relationship, the strength of ζ_8 when the Q($\sqrt[8]{2}$)) is 2. Thus [Q ($\sqrt[8]{2}$, ζ_8): Q] = [Q($\sqrt[8]{2}$, ζ_8): Q($\sqrt[8]{2}$)] [Q($\sqrt[8]{2}$): Q] = 2 · 8 = 16, so | Gal (Q($\sqrt[8]{2}$, ζ_8)/Q) | = 16.

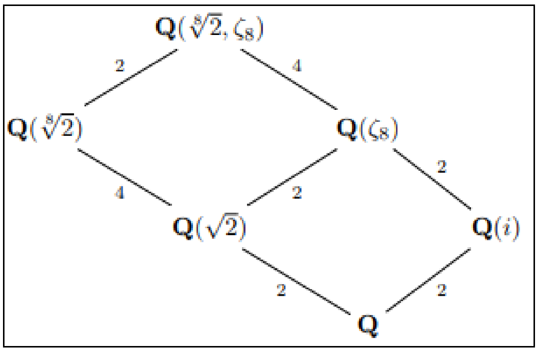
In every σ ∈ Gal (Q($\sqrt[4]{2}$, ζ_8)/Q) depends on the values.

$$\sigma(\zeta_8)=\zeta_8^a \quad (a\in(\mathbf{Z}/8\mathbf{Z})^\times) \quad \text{and} \quad \sigma(\sqrt[8]{2}) = \zeta_8^b\sqrt[8]{2} \quad (b\in\mathbf{Z}/8\mathbf{Z}).$$

A condition is established between a and b in (1.3):

$$\zeta_8^a+\zeta_8^{-a} = \sigma(\zeta_8+\zeta_8^{-1}) = \sigma(\sqrt{2}) = \sigma(\sqrt[8]{2}^4) = \sigma(\sqrt[8]{2})^4 = (-1)^b\sqrt{2}.$$

(This is basically the same as, with the exception that b now originates from Z/8Z rather than Z/4Z.) By using the same logic as in the preceding example, it can be inferred given that b is odd if and only if a ≡ 3, 5 mod 8, as stated in equation (1.6) and even as a result, there are a maximum of four possible values for b for each value of a where a ≡ 1, 7 mod 8. This implies that Gal(Q) is true ($\sqrt[8]{2}$, ζ_8/Q) has no more than 16 orders. The sole limitation is the parity requirement on b for each a, as we have shown that the Galois group has order 16. I am Gal. (Q ($\sqrt[8]{2}$, ζ_8)/Q) This set encompasses the automorphisms supplied by (1.3),4. where (i) b is even when a ≾ 1, 7 mod 8, and (ii) b is odd when a ≍ 3, 5 mod 8.1. Even while it doesn't depict every possible intermediate field, this field diagram does represent several of them.



Every automorphism in Georgia (Q($\sqrt[8]{2}$, ζ_8)/Q), as seen in equation (1.3) $\sigma_{a,b}$ in which $\sigma_{a,b}$ (ζ_8) = ζ_8^a and $\sigma_{a,b}$ ($\sqrt[8]{2}$) = $\zeta_8^b\sqrt[8]{2}$. No less than two automorphisms $\sigma_{a,b}$ and $\sigma_{c,d}$ put out the following:

$$\sigma_{a,b}(\sigma_{c,d}(\zeta_8)) = \sigma_{a,b}(\zeta_8^c) = \zeta_8^{ac},$$

$$\sigma_{a,b}(\sigma_{c,d}(\sqrt[8]{2})) = \sigma_{a,b}(\zeta_8^d\sqrt[8]{2}) = \sigma_{a,b}(\zeta_8)^d\sigma_{a,b}(\sqrt[8]{2}) = \zeta_8^{ad+b}\sqrt[8]{2}.$$

This is how matrices $\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}$ multiply: $\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}c & d\\ 0 & 1\end{pmatrix} = \begin{pmatrix}ac & ad+b\\ 0 & 1\end{pmatrix}$. Therefore Gal (Q($\sqrt[8]{2}$, ζ_8)/Q) possible to see as 2 × 2 mod 8 matrices via

$$\sigma_{a,b} \mapsto \begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}$$

If the requirements are fulfilled, and both a and b belong to the set Z/8Z.
The number b is even for all integers a ≡ 1, mod 8. The value of b is odd for integers a ≡ 3, 5 mod 8.
It must be remembered that this restriction on b given a, or on a certain b, for example, the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has no its effect, since it does not belong to the Galois class $\sigma(\zeta_8) = \zeta_8$ and $\sigma(\sqrt[8]{2}) = (\zeta_8 \sqrt[8]{2})$ Can you tell me which order 16 group what does the Galois group mean? The dihedral group of order 16 is not isomorphic to the Galois group, which has five members of order 2, since it has nine components of order 2, including all eight reflections and the rotation r 4 2: $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 7 & 2 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 7 & 4 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 7 & 6 \\ 0 & 1 \end{pmatrix}$. (The matrices modulo eight $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$ not belong to the Galois group because a matrix of the type $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ with a modulus of 8 and a ≡1, 7 in the Galois group. One component of the Galois set is σ:= $\begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix}$, with an order of 8. The table below shows its successive capabilities. Something that is not a power of σ while it is a member of the Galois group of order two, is τ:= $\begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}$. It is hσ, τ i because order 16 is a property of the Galois group. With the help of a certain mod 8 matrix computation, τστ −1 = τστ = $\begin{pmatrix} 5 & 7 \\ 0 & 1 \end{pmatrix} = \sigma^3$.

k	1	2	3	4	5	6	7	8
σ^k	$\begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 5 & 7 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 5 & 5 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 5 & 3 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

G = x + y, where x is a generator of 8th order and y is a generator of 2nd order, and yx$y^{-1} = x^3$ is named the semi dihedral group of order 16.

$$\mathrm{Gal}(\mathbf{Q}(\sqrt[8]{2}, \zeta_8)/\mathbf{Q}(i)) = \langle \sigma \rangle \cong \mathbf{Z}/8\mathbf{Z}.$$

By establishing which members of the Galois group fix √ 2 = $\sqrt[8]{2^4}$ The reader is expected to demonstrate, given that i is less than or equal to 2,

$$\mathrm{Gal}(\mathbf{Q}(\sqrt[8]{2}, \zeta_8)/\mathbf{Q}(\sqrt{2})) \cong D_4, \qquad \mathrm{Gal}(\mathbf{Q}(\sqrt[8]{2}, \zeta_8)/\mathbf{Q}(i\sqrt{2})) \cong Q_8.$$

It is with this that we end our examination of $Q(\sqrt[8]{2}, \zeta_8)$ on top of the fact that this set's Galois group extends to Q.
After a Galois extension's Galois group has a certain group-theoretic feature-such as being abelian, nonabelian, cyclic, etc.-the extension is referred to as having that characteristic.
An example Any quadratic extension of Q is an abelian extension when the Galois group is of order 2. It is an extension that repeats itself.
Case in point the expansion $Q(\sqrt[3]{2}, \omega)/Q$ is a non-abelian group because it a Galois group that is isomorphic to S3. Considering that field operations are inherently commutative, the word "non-abelian" is irrelevant.
The idea All intermediate fields are abelian extensions of K for any finite abelian extension of L/K. If L/K is cyclic, then any intermediate fields must also be cyclic over K.

The proof. Given that any normal subgroup is an abelian group, it follows that every field F that connects L and K is Galois over K $Gal(F/K) \cong Gal(L/K)/\,Gal(L/F)$. Gal(F/K) is absolutely abelian because the quotient of any abelian group with any subgroup is likewise abelian. It follows that F/K must be cyclic as well if L/K is cyclic, since the quotient of any subgroup with a cyclic group is cyclic.

Conclusion

The analysis of collective behaviors offers a profound insight into the structure and behavior of mathematical objects under symmetry. Burnside's Lemma, in particular, offers a powerful method for counting distinct configurations by eliminating redundancy introduced by symmetries. This technique finds extensive applications in combinatorics, physics, chemistry, and theoretical computer science. We hope that this piece of knowledge has helped you to better grasp how a group's activity on a space's topology and geometry might impact the space's algebraic properties and structure. Geometric group theory is a young and exciting field.

References

1. Raghavan KN. Group Actions and Applications. Chennai: The Institute of Mathematical Sciences; c2012. Available from: knr@imsc.res.in.
2. Spieler RC. Group Actions; c2014.
3. Pap EJ, Waalkens H. Frames of Group Sets and Their Application in Bundle Theory. Mathematics. 2024;12(13):2135.
4. Tang D, Wang Z, Yue B. Applications of Group Theory. In Journal of Physics: Conference Series 2022;2381(1):012110. IOP Publishing.
5. Orgest ZA. Image understanding and applications of symmetry groups. Journal Of Algebra And Computional Applications. 2011;1(1):20-30.
6. Gromov M. Asymptotic invariants of infinite groups. In: Geometric Group Theory. Vol. 2. Cambridge: Cambridge University Press; c1993. p. 1–295.
7. Italiano G, Martelli B, Migliorini M. Hyperbolic 5-manifolds that fiber over S^1. Preprint. arXiv:2105.14795.
8. Dummit DS, Foote RM. Abstract Algebra. 2nd ed. New York: John Wiley & Sons; c1999.
9. Rotman JJ. A First Course in Abstract Algebra. 2nd ed. Upper Saddle River (NJ): Prentice Hall; c2000.
10. Ash RB. Abstract Algebra: The Basic Graduate Year. Available from: http://www.math.uiuc.edu/~r-ash/Algebra.html.