E-ISSN: 2583-9667
Indexed Journal
Peer Reviewed Journal
https://multiresearchjournal.theviews.in

Received: 10-12-2025
Accepted: 08-01-2026
Published: 15-01-2026

INTERNATIONAL JOURNAL OF ADVANCE RESEARCH IN MULTIDISCIPLINARY

Volume 4; Issue 1; 2026; Page No. 162-165

Cantor set, Perfect sets and its Uncountability

Dr. BS Satpute

Head, Department of Mathematics, Vaidyanath College, Parali, Beed, Maharashtra, India

DOI: https://doi.org/10.5281/zenodo.19116349

Corresponding Author: Dr. BS Satpute

Abstract

In this paper I give a brief review about Cantor set, Perfect sets and its Uncountability.

Keywords: Cantor set, Uncountability, Continuum Hypothesis, brief review, perfect sets

Introduction

Now days descriptive set theory is understood as the study of definable subsets of a special topological spaces Polish Spaces. Many of its problems and techniques are results of efforts to answer the basic questions about the real numbers. An important example is the *Continuum Hypothesis* (CH): If $A \subseteq \mathbb{R}$ is uncountable, does there exist a bijection between A and $\mathbb{R}$? That is, is every uncountable subset of $\mathbb{R}$ is of the same cardinality as that of $\mathbb{R}$? [Cantor, 1890's].

Initially this problem was tried to solve to show that CH holds for a number of sets with an easy topological structure. It is a very standard exercise to show that every open set and closed satisfies CH. That is an open set and closed sets contains an interval, which maps bijectively to $\mathbb{R}$.

In this article we consider the set of real numbers $\mathbb{R}$ as a universal set and $\mathbb{N}$ = {1,2,3,…} the set of natural numbers $\mathbb{Z}$ = {... –3, - 2, -1,0,1,2…}, the set of integers $\mathbb{Q} = \left\{\frac{p}{q} : p, q \neq 0 \in \mathbb{Z}\right\}$, as set of rational numbers

A set S is a finite set if S is either empty or there is one-to-one correspondence between S and {1,2,3,…,n} the set containing first n natural numbers for some $n \in \mathbb{N}$.

Approximating irrational numbers: Let us starts with an irrational number β. An approximation of β is defined as a sequence of rational numbers which converging to β. We know the notions of the speed of an approximation and the distance of a given number from the rational numbers. These ideas can relate the growth rate of the denominators of an approximation and rate of its convergence.

A set S is said to be countably infinite if there is a one-to-one correspondence between S and $\mathbb{N}$. A set S is said to be countable if it is a finite or countably infinite set. A set which is not countable is called an uncountable set. Subset of countable sets is countable and subset of uncountable sets may be uncountable.

Isolated points: A point x of a set D is called an isolated point if there is a neighborhood of x which does not contain any point of D other than x, that is x is not a limit point of D.

There are some divergences in terminology. Some authors call a limit point an accumulation point.

Perfect set: A subset P of $\mathbb{R}$ is said to be perfect if it is closed set and its every point is its limit point.

We can easily see that for a perfect set P, every NBD of a point in P contains infinitely many points.

For example for any a < b in $\mathbb{R}$, [a,b] is a nonempty perfect set. $\mathbb{R}$ itself is a perfect set. A nonempty perfect set is uncountable.

Finite union of perfect sets is a perfect set, but arbitrary union of perfect sets need not be a perfect set.

For example, for each $n \in \mathbb{N}$, $\left[2+\frac{1}{n}, 5-\frac{1}{n}\right]$ is a perfect set

But $\bigcup_{n=1}^{\infty}\left[2-\frac{1}{n}, 5-\frac{1}{n}\right] = (2,5)$ is not a perfect set.

There are totally disconnected perfect sets, such as the middle-third Cantor set in [0, 1]

Theorem: The cantor set C is a perfect.

Proof: We know that each C_n is finite union of closed intervals. Therefore each C_n is closed. We have $C = \cap C_n$, the cantor, and therefore C is closed.

Now let us show that $x \in C$ is not isolated

We will show it by constructing a sequence (x_n) in C such that $f(x_n) = x$ for all $n \in \mathbb{N}$ and $x_n \to x$.

The closed set C_1 is union of closed intervals I_{11} and I_{21} with length $\frac{1}{3}$.

Then $x \in I_1$ or I_2. Call it is belongs to I_1. $C_2 \cap I_1$ consists of two closed intervals, one of them contains x. Pick up x_1 an endpoint of other closed interval. Therefore $f(x_1) = x$.

We have $x_1 \in C$, $x, x_1 \in I_1 \Rightarrow |x_1 - x| \le \frac{1}{3}$,

The closed set C_2 is the union of four intervals $I_{21}, I_{22}, I_{23}, I_{24}$ which all are closed and length of each interval is $\frac{1}{9}$.

Then $x \in I_{21} or\ I_{22}\ or\ I_{23}\ or\ I_{24}$. Call it is belongs to I_2. $C_3 \cap I_2$ Consists of two closed intervals, one of them contains x. Pick up x_2 an endpoint of other closed interval. Therefore $f(x_2) = x$.

We have $x_2 \in C$, since x_2 is endpoint of one of the interval in C_3.

Now x_2 and x both belongs to I_2 which implies $|x_2 - x| \le \frac{1}{9}$.

Continuing in this way, we construct a sequence (x_n) in C such that $f(x_n) = x$ for all $n \in \mathbb{N}$ and $|x_2 - x| \le \frac{1}{3^n}$

Thus x is a limit point of C, therefore x is not isolated.

Therefore C is perfect.

Remark: Her we have constructed a sequence (x_n) in C_n by using endpoints of closed intervals in C_n and each endpoint is a rational point of the form $\frac{m}{3^n}$ $0 \le m \le 3^n$, but it doesn't mean that it converges to rational point.

In fact in most of the sequences constructed from rational points converges to irrational points and this account to the uncountable nature of Cantor set C.

Theorem: A nonempty perfect set is uncountable.

Proof: Note that a nonempty perfect set is never finite, since in a finite set each point is isolated.

Assume contrary that a perfect set P is countably infinite.

Now we can enumerate the elements of P by using the bijection from $\mathbb{N}$ onto P as

$$P = \{x_1, x_2, x_3, \dots\}$$

Let x_1 be an interior point of the closed interval I_1.

The element x_1 is not isolated, so there a y_2 in P such that y_2 is also interior point of I_1. Choose a closed interval I_2 centered on y_2 so that $I_2 \subseteq I_1$ and $f(x) \in I_1$. Here $I_2 \cap P \ne \phi$, since y_2 is both in $I_2 and\ P$. The element y_2 is not isolated, therefore there is y_3 in P and in the interior of I_2.

Choose $y_3 = f(x_2)$, for if $y_3 = x_2$ then there will be another choice of y_3 in the interior of I_2 and in P because y_3 is not isolated.

Now choose a closed interval $I_3 \subseteq I_2$ centered at y_3 for which $f(x_2) \in I_3$. Here $y_3 \in I_3 \cap P$, therefore $I_3 \cap P \ne \phi$.

Continuing this construction inductively results in a sequence of closed intervals I_n satisfying following conditions,

- $I_n \subseteq I_{n-1}$,
- $f(x_n) \in I_{n+1}$,
- $I_n \cap P \ne \phi$

Here $K_n = I_n \cap P$, $n \in \mathbb{N}$, is compact.
Therefore we have that $\cap_1^\infty K_n = \phi$ which is contradiction
So our assumption that P is countable is a contradiction.
Which implies a perfect set is uncountable.

Theorem 1.1: Any perfect subset of $\mathbb{R}$ has the same cardinality as $\mathbb{R}$.

Proof. Let $P \subseteq \mathbb{R}$ R be any perfect subset of $\mathbb{R}$. We will construct an injection from the set $2^{\mathbb{N}}$ of all infinite binary sequences into P. An infinite binary sequence $\xi = \xi_0\xi_1\xi_2 \cdots$ can be identified with $r \in [0,1]$ by defining the mapping.

$$\xi \to \sum_{i \geq 0} \xi_i 2^{-i-1}$$

This mapping is onto. Therefore the cardinality of P is at least as large as the that of [0,1]. The Cantor-Schroder-Bernstein Theorem $|\boldsymbol{P}| = 2^{\mathbb{N}}$(for a Proof see e.g. Jech [2003]).

Choose $x \in P$, and $\xi_0 = 1 = 2^0$. As P is perfect, $P \cap U_{\varepsilon_0}(x)$. Let $x_0 \neq x_1$ be two points in $P \cap U_{\varepsilon_0}(x)$, distinct from x. Let ε_1 be such that $2^{-\varepsilon_1} \leq \frac{1}{2}$, $U_{\varepsilon_0}(x_0), U_{\varepsilon_1}(x_1) \subseteq U_{\varepsilon_0}(x)$, and $\overline{U_{\varepsilon_0}(x_0)} \cap \overline{U_{\varepsilon_1}(x_1)}$ where $\overline{U}$ denotes the closure of U. Iterating this procedure recursively with smaller and smaller diameters we obtain a family of open balls (U_σ) by using the fact that *P* is perfect. This is a so-called Cantor scheme. Here the index σ is a finite binary sequence which is also called a *string*. Which has the following properties,

- C1) diam$(U_\sigma) \leq 2^{-|\sigma|}$, $|\sigma|$ denotes the length of σ.
- C2) If τ *and* σ are incompatible (i.e. neither extends the other), then $U_\tau \cap U_\sigma = \phi$.
- C3) The center of every U_σ is in P, call it x_σ.

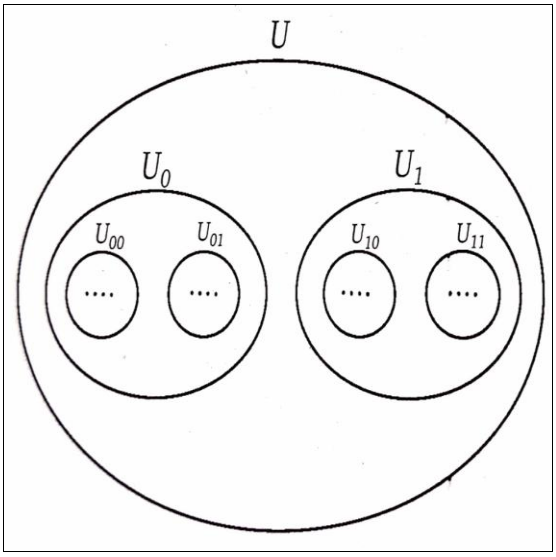

Fig 1: Cantor Scheme

Let ξ be an infinite binary sequence. We denote the string formed by first n bits of ξ by $\xi|_n$ for given $n \geq 0$.
i.e. $\xi|_n = \xi_0\xi_1\xi_2 \cdots \xi_{n-1}$

The finite initial segments give rise to a sequence $x_{\xi|_n}$ of centers. This is a Cauchy sequence, by (C1) and (C2) and by (C4), the sequence lies in *P*. *P* is

Closed and the limit x_ξ is in *P*. By (C3), the mapping $\xi \to x_\xi$ is well-defined and injective.

Theorem 1.2: Every uncountable closed subset of $\mathbb{R}$ contains a perfect subset.

Proof. Let $C \subseteq \mathbb{R}$ be an uncountable closed subset of $\mathbb{R}$. Say $z \in \mathbb{R}$ is a condensation point of C.
$\forall \epsilon > 0[U_\epsilon(z) \cap C$ uncountable]

Let D be the set of all condensation points of C. Every condensation point is clearly an accumulation point and C is closed. We claim that D is perfect. Clearly D is closed. Suppose $z \in D$ and $\epsilon > 0$. Then $U_\epsilon(z) \cap C$ is uncountable. We would like to conclude that $U_\epsilon(z) \cap D$ is uncountable, too, since this would mean in particular that $U_\epsilon(z) \cap D$ is infinite. It holds if $C - D$ is countable. To show that $C - D$ is countable, let us use the fact that every open interval in $\mathbb{R}$ is the union of countably many open intervals with rational endpoints. Note that there are only countably many such intervals. If $y \in C - D$, then for some $\delta > 0$, $U_\delta(y) \cap C$ is countable. y is contained in some subinterval $U_y \subseteq U_\delta(y)$ with rational endpoints. Thus, we have

$$C - D \subseteq \bigcup_{y \in C - D} U_y \cap C.$$

Here the right-hand side is a countable union of countable sets and therefore it is countable.

Corollary 1.3: Every closed subset of R is either countable or of the cardinality of the continuum.

References

1. Gorkem. Nonempty perfect subset of R that contains no rational numbers. 2013 Mar 21.
2. Holland J. Math 1530: Homework 4 solution. 2008 Oct 6.
3. Mansfield R. Perfect subsets of definable sets of real numbers. Pacific Journal of Mathematics. 1970;35(2).
4. Royden HL. Real analysis. 2nd Ed. New York: Macmillan; 1968.
5. Rudin W. Principles of mathematical analysis. 3rd Ed. New York: McGraw-Hill; c1976.