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Completeness of R^m

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Abstract

Using the result ‘every sequence of real numbers has a monotone subsequence’ we prove: If a bounded sequence has unequal lim sup and lim inf, then it has at least two monotone (convergent) subsequences whose range sets are disjoint; (BW theorem) Every bounded sequence has a convergent subsequence, and R is a complete space under usual metric on R. From these we obtain easy proofs of (BW theorem): Every bounded sequence in a Euclidean space has a convergent subsequence, and every Euclidean space is complete.

Keywords: Euclidean space, Completeness of R^m, two monotone, subsequences, BW theorem

Introduction

Sequences of real numbers: A sequence is a function whose domain is N = {1, 2, 3,...}. If s: N ⟶ K (= R or C) then s is a sequence in K and it is denoted by $\{s_n\}_{n=1}^{\infty}$ or $\{s_n\}$ or $< s_n >$ or (s_n). If $s_n \in R$ ∀ n then $\{s_n\}$ is a sequence of real numbers etc. Range of sequence is $\{s_1, s_2, s_3,...\}$.

LUB Axiom: If A is a nonempty subset of R which is bounded above has the least upper bound in R, i.e. lub A ∈ R. Using lub axiom we have; GLB Property: If B is a nonempty subset of R which is bounded below has greatest lower bound in R, i.e. glb A ∈ R.

In this section we consider sequences in R. A sequence $\{s_n\}$ is bounded means its range is bounded, i.e. there is a number M > 0 such that $|s_n| \le M$ for all n ∈ N. *A number l is called a *limit* of the sequence iff for any ε > 0, ∃ n_0 ∈ N such that ∀ $n \ge n_0 \Rightarrow |s_n - l| < \varepsilon$. In this case we write $\lim_{n\to\infty} s_n = l$ or $\lim s_n = l$ or $s_n \longrightarrow l$ as n ⟶ ∞, and we say that $\{s_n\}$ is a *convergent sequence* (*converges to* l). If a sequence is not convergent, then it is said to be *divergent* (oscillatory or properly divergent). $\lim_{n\to\infty} s_n = +\infty$ (sequence $\{s_n\}$ diverges to +∞) means for any M > 0 (however large), ∃ n_0 ∈ N

such that ∀ $n \ge n_0 \Rightarrow s_n > M$. $\lim_{n\to\infty} s_n = -\infty$ (sequence $\{s_n\}$ diverges to –∞) means for any M > 0 (however large), ∃ n_0 ∈ **N**

such that ∀ $n \ge n_0 \Rightarrow s_n < -M$. A sequence $\{s_n\}$ is said to be *properly divergent* if either $\lim_{n\to\infty} s_n = \infty$ or $\lim_{n\to\infty} s_n = -\infty$. A sequence which is neither convergent nor properly divergent is said to be *oscillatory*..

Theorem

If a limit of a sequence exists then it is unique.

Theorem

If a sequence $\{s_n\}$ of real (or complex) numbers is convergent, then it is bounded.

Proof: Let $\{s_n\}$ be a convergent sequence and its limit is a number c. Then for ε = 1, ∃ m ∈ **N** such that $n \ge m \Rightarrow ||s_n| - |c|| < |s_n - c| < 1$. $\Rightarrow |s_n| < 1 + |c|$ ∀ n and hence the sequence $\{s_n\}$ is bounded.

Algebra of limits and some theorems

Let $< c > = (c, c, c, \ldots)$ be a constant sequence and $\lim_{n\to\infty} s_n = l$, $\lim_{n\to\infty} t_n = m$. Then

- $\lim_{n\to\infty} c = c$.
- $\lim_{n\to\infty} [s_n \pm t_n] = l \pm m$.
- $\lim_{n\to\infty} cs_n = c\,l$.
- $\lim_{n\to\infty} s_n^k = l^k$ for $k \in \mathbf{N}$.
- $\lim_{n\to\infty} s_n t_n = l m$.
- $\lim_{n\to\infty} \frac{1}{t_n} = \frac{1}{m}$, provided $m \neq 0$.
- $\lim_{n\to\infty} \frac{s_n}{t_n} = \frac{l}{m}$ provided $m \neq 0$.
- $\lim_{n\to\infty} s_{n+p} = l$ for any fixed $p \in \mathbf{N}$.
- $\lim_{n\to\infty} |s_n| = |l|$.
- $\lim_{n\to\infty} s_n = 0$ iff $\lim_{n\to\infty} |s_n| = 0$.
- If $s_n \geq 0\ \forall\, n \in N$ (or $\forall\, n \geq n_0$ for some fixed $n_0 \in N$) $\Rightarrow l \geq 0$. (xii) $s_n \leq t_n\ \forall\, n \geq n_0$, $n_0 \in N$ is fixed $\Rightarrow l \leq m$.
- **Sandwich Theorem:** If $s_n \leq u_n \leq v_n\ \forall\, n \in N$ (or $\forall\, n \geq n_0$ for some fixed $n_0 \in N$) and $\lim_{n\to\infty} s_n = l$, $\lim_{n\to\infty} v_n = l$ then $\lim_{n\to\infty} u_n = l$. (xiv) $\lim_{n\to\infty} a^n = 1$, for any fixed real number $a > 0$.

Monotonic sequences: Let $\{s_n\}$ be a sequence of real numbers. It is called

- A monotonically increasing (m. i.) if $s_n \leq s_{n+1}\ \forall\, n \in N$.
- Strictly increasing if $s_n < s_{n+1}\ \forall\, n \in N$.
- Monotonically decreasing (m. d.) if $s_n \geq s_{n+1}\ \forall\, n \in N$.
- Strictly decreasing if $s_n > s_{n+1}\ \forall\, n \in N$.

Note also that increasing = non-decreasing etc and a m. i. sequence is bounded below by its first term and a m. d. sequence is bounded above by its first term.

A sequence is a constant sequence iff it is m. i. as well as m. d. A sequence which is m. i. or m. d. is called a *monotone sequence*.

Theorem

Let $\{s_n\}$ be a m. i. sequence. If $\{s_n\}$ is bounded then it converges to $\sup_n s_n$, otherwise diverges to $+\infty$.

Proof: Let $\{s_n\}$ be a m. i. bounded sequence. Then the range $\{s_1, s_2, s_3, \ldots\}$ of the sequence is bounded above and by lub axiom, its lub $l \in R$, i.e. $\ell = \text{lub}\,\{s_1, s_2, s_3, \ldots\} = \sup_{n \geq 1} s_n$.

For any $\varepsilon > 0$, $l - \varepsilon$ is not an upper bound of $\{s_1, s_2, \ldots\}$, so $\exists\, n_0 \in N$ such that $l - \varepsilon < s_{n_0}$.

But $s_{n_0} \leq s_{n_0+1} \leq s_{n_0+2} \leq \ldots \leq l < l + \varepsilon$. Hence $\forall\, \varepsilon > 0$, $\exists\, n_0 \in N$ such that

$n \geq n_0 \Rightarrow l - \varepsilon < s_n < l + \varepsilon$, i.e. $|s_n - l| < \varepsilon$. This proves $\lim_{n\to\infty} s_n = l = \sup s_n$.

If $\{s_n\}$ is not bounded above, then for any $M > 0$, $\exists\, n_0 \in N$ such that $s_{n_0} > M$.

Then $\forall\, n \geq n_0 \Rightarrow s_n \geq s_{n_0} \Rightarrow s_n > M$, i.e. $\lim_{n\to\infty} s_n = +\infty$.

Corollary 1

Let $\{s_n\}$ a m. d. sequence. If $\{s_n\}$ is bounded then it converges to $\inf_n s_n$, otherwise diverges to $-\infty$.

Subsequence

If $\{k_n\}_{n=1}^{\infty}$ is any strictly increasing sequence of positive integers (i.e. $k_n < k_{n+1}$ in **N** and $k_n \geq n\ \forall\, n \in \mathbf{N}$) then $\{s_{k_n}\}_{n=1}^{\infty}$ is called a *subsequence* of the sequence $\{s_n\}_{n=1}^{\infty}$.

Theorem

If $\lim\limits_{n\to\infty} s_n = l$ (or $\pm\infty$) then any subsequence of $\{s_n\}$ converges to l (or $\pm\infty$).

Proof: (i) Let $\lim\limits_{n\to\infty} s_n = l$. Then $\forall\ \varepsilon > 0, \exists\ n_0 \in \mathbf{N}$ such that $n \geq n_0 \Rightarrow |s_n - l| < \varepsilon$.

Let $\{s_{k_n}\}_{n=1}^{\infty}$ be any subsequence of $\{s_n\}$. Then $n \geq n_0 \Rightarrow k_n \geq n \geq n_0 \Rightarrow |s_{k_n} - l| < \varepsilon$.

Hence $\lim\limits_{n\to\infty} s_{k_n} = l$. (ii) Let $\lim\limits_{n\to\infty} s_n = \infty$. Then $\forall\ M > 0$ (however large) $\exists\ n_0 \in \mathbf{N}$ such that $n \geq n_0 \Rightarrow s_n > M$

$\Rightarrow s_{k_n} > M$ for any subsequence $\{s_{k_n}\}$ of $\{s_n\} \Rightarrow \lim\limits_{n\to\infty} s_{k_n} = \infty$.

From above theorem (by contra positive): If a sequence has two subsequences with different limits then the sequence is not convergent.

Theorem

$\lim\limits_{n\to\infty} s_n = l\ (\pm\infty)$ iff $\lim\limits_{n\to\infty} s_{2n} = \lim\limits_{n\to\infty} s_{2n-1} = l\ (\pm\infty)$.

Proof: Let $\lim\limits_{n\to\infty} s_n = \ell$, i.e. $\{s_n\}$ is a convergent sequence and its limit is $l \in \mathbf{R}$.

Then every subsequence of $\{s_n\}$ also converges to l. In particular $\lim\limits_{n\to\infty} s_{2n} = \lim\limits_{n\to\infty} s_{2n-1} = l$.

Conversely let $\lim\limits_{n\to\infty} s_{2n} = \lim\limits_{n\to\infty} s_{2n-1} = l$. Then $\forall\ \varepsilon > 0, \exists\ n_0 \in \mathbf{N}$ such that

$N \geq n_0 \Rightarrow |s_{2n} - l| < \varepsilon$ and $|s_{2n-1} - l| < \varepsilon$. (*)

Consider any integer $m \geq 2n_0$. If m is even, say $m = 2n$, then $2n \geq 2n_0 \Rightarrow n \geq n_0$ and if m is odd, say $m = 2n - 1$, then $2n - 1 \geq 2n_0 \Rightarrow n > n_0$, so by (*) we have $|s_m - l| < \varepsilon\ \forall\ m \geq 2n_0 - 1$.

Hence $\lim\limits_{m\to\infty} s_m = l$, i.e. $\lim\limits_{n\to\infty} s_n = l$.

Cauchy sequence

A sequence $\{s_n\}$ is called a *Cauchy sequence* iff $\forall\ \varepsilon > 0, \exists\ n_0 \in N$ such that $m, n \geq n_0 \Rightarrow |s_m - s_n| < \varepsilon$, i.e. $n \geq n_0 \Rightarrow |s_{n+p} - s_n| < \varepsilon\ \forall\ p \in N$.

Results

In R (or C), a sequence is Cauchy iff it is convergent. Every convergent sequence is bounded but not conversely.

For example $\{(-1)^{n-1}\}_{n=1}^{\infty} = (1, -1, 1, -1, 1, -1, \ldots)$ is bounded, has two subsequences $(1, 1, 1, \ldots) \longrightarrow 1$, $(-1, -1, -1, \ldots) \longrightarrow -1$ with different limits and hence the sequence is not convergent.

Limit points: A number p is said to be a *limit* (*cluster*) *point* of a sequence $\{s_n\}$ of every nbd of p contains infinitely many terms of the sequence.

In this case $\forall\ \varepsilon > 0$, $s_n \in (p - \varepsilon, p + \varepsilon)$ for infinitely many values of n. If c is the limit of a sequence $\{s_n\}$ then c is the only limit point of $\{s_n\}$. For a bounded sequence of real numbers $\underline{\lim}\ s_n$ and $\overline{\lim\limits_{n\to\infty}}\ s_n$ are limit points, which are the least and greatest. The set of limit points of a bounded sequence is bounded and it has the greatest and least limit points. The set of limit points of a sequence is a closed set and hence the set of limit points of a bounded sequence is a compact set.

Theorem (*Bolzano – Weierstrass Theorem*): Every bounded sequence has a convergent subsequence.
BW Theorem also stated as: Every bounded sequence has at least one limit point.

Theorem

Every Cauchy sequence of real numbers is bounded.

Proof: Let $\{s_n\}$ be a Cauchy sequence. Then for $\varepsilon = 1$, $\exists$ k $\in$ N such that
$m, n \geq k \Rightarrow |s_n - s_m| = |s_n - s_k + s_k| \leq |s_n - s_k| + |s_k| < 1 + |s_k|\ \forall\ n \geq k$.
Let $M = \max \{|s_1|, |s_2|, \dots, |s_{k-1}|, 1 + |s_k|\}$. Then $|s_n| \leq M\ \forall\ n \in \mathbf{N}$, i.e. $\{s_n\}$ is a bounded sequence.

Theorem

Cauchy's convergence criterion: A sequence $\{s_n\}$ of real numbers converges iff $\{s_n\}$ is a Cauchy sequence, i.e. $\forall\ \varepsilon > 0\ \exists\ n_0 \in N$ such that $m, n \geq n_0 \Rightarrow |s_m - s_n| < \varepsilon$.
Above theorem shows that R is a complete metric space under the usual metric.

Limit superior and limit inferior

Let $\{s_n\}$ be a sequence of real numbers and for $n \in N$, let $a_n = \text{glb}\ \{s_n, s_{n+1}, s_{n+2}, \dots\} = \inf_{k\geq n} s_k$,
$b_n = \text{lub}\ \{s_n, s_{n+1}, s_{n+2}, \dots\} = \sup_{k\geq n} s_k$. Then $a_n \leq s_n \leq b_n\ \forall\ n \in N$.

$\{a_n\}$ is m. i. and bounded above by b_1 (bounded below by a_1), so $\lim_{n\to\infty} a_n = \sup_{n\geq 1} a_n \in R$ or $\lim_{n\to\infty} a_n = \pm\infty$ ($\lim_{n\to\infty} a_n = -\infty$ if $a_n = -\infty\ \forall$ n).

$\{b_n\}$ is m. d. and bounded below by a_1 (bounded above by b_1), so $\lim_{n\to\infty} b_n = \sup_{n\geq 1} b_n \in R$ or

$\lim_{n\to\infty} b_n = \pm\infty$ ($\lim_{n\to\infty} b_n = \infty$ if $b_n = \infty\ \forall$ n). Also note that $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

$\lim_{n\to\infty} a_n$ is called a *lower limit* or *limit inferior* of the sequence $\{s_n\}$ and it is denoted by $\liminf_{n\to\infty} s_n$ or $\underline{\lim}\ s_n$.

$\lim_{n\to\infty} b_n$ is called an *upper limit* of the sequence $\{s_n\}$ and it is denoted by $\limsup_{n\to\infty} s_n$ or $\overline{\lim}\ s_n$.

Note that $\lim_{n\to\infty} s_n = \ell\ (\pm\infty)$ iff $\underline{\lim}\ s_n = \overline{\lim}\ s_n = \ell\ (\pm\infty)$ iff $\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n-1} = \ell\ (\pm\infty)$.

Limit superior, limit inferior of a sequence of real numbers always exist in the extended real number system.

Note that $\underline{\lim}\ s_n = \lim_{n\to\infty} a_n = \sup_{n\geq 1} a_n = \sup_{n\geq 1} \inf_{k\geq n} s_k$ and $\overline{\lim}\ s_n = \inf_{n\geq 1} \sup_{k\geq n} s_k$.

For $s_n \in R\ \forall$ n, let $a = \lim_{n\to\infty} s_{2n-1}$, $b = \lim_{n\to\infty} s_{2n}$ (if exist) then $\underline{\lim}\ s_n = \min \{a, b]$, $\overline{\lim}\ s_n = \max \{a, b\}$.

Results

- A real number u is the limit superior of a bounded sequence $\{s_n\}$ iff (i) $\forall\ \varepsilon > 0$, $s_n > u - \varepsilon$ for infintely many values of n and (ii) $\forall\ \varepsilon > 0$, $s_n < u + \varepsilon$ for all except finitely many values of n.
- A real number ℓ is the limit inferior of a bounded sequence $\{s_n\}$ iff (i) $\forall\ \varepsilon > 0$, $s_n < \ell + \varepsilon$ for infintely many values of n and (ii) $\forall\ \varepsilon > 0$, $s_n > \ell - \varepsilon$ for all except finitely many values of n.
- A sequence $\{s_n\}$ converges to ℓ iff $\limsup s_n = \liminf s_n = \ell$.

Theorem

Every sequence contains a monotonic subsequence.

Proof: If a sequence $\{s_n\}_{n=1}^{\infty}$ has a term c repeated infinitely, then it has a constant subsequence (c, c, c,...) of the sequence $\{s_n\}_{n=1}^{\infty}$, which is monotone. Next consider $\{s_n\}_{n=1}^{\infty}$ as a sequence which has infitely many distinct terms and no term is infinitely repeated. Let $U = \limsup s_n$ and $L = \liminf s_n$ be extended real numbers.

For $U = \infty$, choose $k_1 \in N$ such that $s_{k_1} > 1$. Choose least integer $k_2 > k_1$ such that $s_{k_2} > \max \{2, s_{k_1}\}$

Since $s_n \in (\max \max \{2, s_{k_1}\}, \infty)$ for infinitely many values of n, as $\limsup s_n = \infty$. Next choose least integer integer $k_3 > k_2$

such that s_{k_3} > max {3, s_{k_2}} and so on. In this way we obtain a strictly increasing subsequence $\{s_{k_n}\}_{n=1}^{\infty}$ of $\{s_n\}_{n=1}^{\infty}$ such that s_{k_n} > n ∀ n ∈ N; showing $\lim_{n\to\infty} s_{k_n} = \infty$.

For L = $-\infty$, we have lim sup ($-s_n$) = – lim inf s_n = $-L$ = ∞, so the sequence $\{-s_n\}_{n=1}^{\infty}$ has strictly increasing subsequence $\{-s_{k_n}\}_{n=1}^{\infty}$ diverging to + ∞. ⇒ $\{s_n\}_{n=1}^{\infty}$ has strictly decreasing subsequence $\{s_{k_n}\}_{n=1}^{\infty}$ diverging to $-\infty$. For U ∈ R, s_n ∈ (U – ε, U) for infinitly may values of n or s_n ∈ (U, U + ε) for infinitely many values of n. Consider s_n ∈ (U – ε, U) for ifinitly may values of n and let A = {s_n | s_n ∈ (U – ε, U)}. For any s_m ∈ A,

take s_m = s_{k_1}, i.e. k_1 = m. Choose least k_2 ∈ N, $k_2 > k_1$ and s_{k_2} ∈ A with $s_{k_2} \geq s_{k_1}$. Next choose least k_3 ∈ N, $k_3 > k_2$ and s_{k_3} ∈ A with $s_{k_3} \geq s_{k_2}$ and so on. In this way we obtain a monotonic increasing subsequence $\{s_{k_n}\}_{n=1}^{\infty}$ of $\{s_n\}_{n=1}^{\infty}$ which converges to U. Similarly if s_n ∈ (U, U + ε) for infinitely many values of n, we obtain a monotonic decreasing subsequence of $\{s_n\}_{n=1}^{\infty}$ converging to U.

For L ∈ R, s_n ∈ (L – ε, L) for infinitly may values of n or s_n ∈ (L, L + ε) for infinitely many values of n. Consider s_n ∈ (L, L + ε) for ifinitly may values of n and let B = {s_n | s_n ∈ (L, L + ε)}. For any s_m ∈ B,

Take s_m = s_{t_1}, i.e. t_1 = m. Choose least t_2 ∈ N, $t_2 > t_1$ and s_{t_2} ∈ A with $s_{t_2} \leq s_{t_1}$. Next choose least t_3 ∈ N, $t_3 > t_2$ and s_{t_3} ∈ B with $s_{t_3} \leq s_{t_2}$ and so on. In this way we obtain a monotonic decreasing subsequence $\{s_{k_n}\}_{n=1}^{\infty}$ of $\{s_n\}_{n=1}^{\infty}$ which converges to L. Similarly if s_n ∈ (L – ε, L) for infinitely many values of n, we obtain a monotonic increasing subsequence of $\{s_n\}_{n=1}^{\infty}$ converging to L.

Thus every sequence has a monotone subsequence.

Theorem: If a bounded sequence of real numbers has unequal lim sup and lim inf then it has atleast two monotone (convergent) subsequences whose range sets are disjoint.

Proof: Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Then lim inf s_n = L, lim sup s_n = U are real numbers and L ≤ U.

Consider L ≠ M, i.e. L < M. Then ε = $\frac{M-L}{3}$ > 0 is a real number.

(I) If L appears infinitely as terms of the sequence $\{s_n\}_{n=1}^{\infty}$ then $\{s_{k_n}\}_{n=1}^{\infty}$ = (L, L, L,...) is a monotone subsequence (converging to L) of the sequence $\{s_n\}_{n=1}^{\infty}$ and s_{k_n}∈ (L – ε, L + ε) ∀ n ∈ **N**.

Next consider that L appears as terms of the sequence $\{s_n\}_{n=1}^{\infty}$ at most a finite number. Then s_n ∈ (L – ε, L) for infinitely many values of n, in which case the sequence $\{s_n\}_{n=1}^{\infty}$ has a monotone increasing subsequence $\{s_{k_n}\}_{n=1}^{\infty}$ converging to L and s_{k_n}∈ (L – ε, L) ∀ n ∈ N; or s_n ∈ (L, L + ε) for infinitely many values of n, in which case the sequence $\{s_n\}_{n=1}^{\infty}$ has a monotone decreasing subsequence $\{s_{k_n}\}_{n=1}^{\infty}$ converging to L and s_{k_n}∈ (L, L + ε) ∀ n ∈ N. Thus $\{s_n\}_{n=1}^{\infty}$ has a monotone subsequence $\{s_{k_n}\}_{n=1}^{\infty}$ converging to L and s_{k_n}∈ (L – ε, L + ε) ∀ n ∈ N.

(II) If U appears infinitely as terms of the sequence $\{s_n\}_{n=1}^{\infty}$ then $\{s_{k_n}\}_{n=1}^{\infty}$ = (U, U, U,...) is a monotone subsequence (converging to U) of the sequence $\{s_n\}_{n=1}^{\infty}$ and s_{k_n}∈ (U – ε, U + ε) ∀ n ∈ N. Next consider that U appears as terms of the sequence $\{s_n\}_{n=1}^{\infty}$ at most a finite number. Then s_n ∈ (U – ε, U) for infinitely many values of n, in which case the sequence $\{s_n\}_{n=1}^{\infty}$ has a monotone increasing subsequence $\{s_{t_n}\}_{n=1}^{\infty}$ converging to U and s_{t_n}∈ (U – ε, U) ∀ n ∈ N; or s_n ∈ (U, U + ε) for infinitely many values of n, in which case the sequence $\{s_n\}_{n=1}^{\infty}$ has a monotone decreasing subsequence $\{s_{t_n}\}_{n=1}^{\infty}$ converging to U and s_{t_n}∈ (U, U + ε) ∀ n ∈ N.

Thus $\{s_n\}_{n=1}^{\infty}$ has a monotone subsequence $\{s_{t_n}\}_{n=1}^{\infty}$ converging to U and s_{t_n}∈ (U – ε, U + ε) ∀ n ∈ **N**. We have $\{s_{k_n}\}_{n=1}^{\infty}$ is a monotone (convergent) subsequence of $\{s_n\}_{n=1}^{\infty}$ in (L – ε, L + ε), $\{s_{t_n}\}_{n=1}^{\infty}$ is a monotone (convergent) subsequence of $\{s_n\}_{n=1}^{\infty}$ in (U – ε, U + ε) and (L – ε, L + ε)∩(U – ε, U + ε) = $\emptyset$. So we have at least two monotone subsequences $\{s_{k_n}\}_{n=1}^{\infty}$ and $\{s_{t_n}\}_{n=1}^{\infty}$ of the sequence $\{s_n\}_{n=1}^{\infty}$ whose range sets are disjoint.

Note: From Theorem 1.9

(1) Every bounded sequence of real numbers has a monotone (and bounded) subsequence, which is convergent. Limit of this convergent subsequence, which is a real number, is a limit point of the sequence. From this BW theorem follows.

(2) Consider any Cauchy sequence $\{s_n\}_{n=1}^{\infty}$ of real numbers. Then $\{s_n\}_{n=1}^{\infty}$ is a bounded sequence and hence it has a bouded monotone subsequence $\{s_{k_n}\}_{n=1}^{\infty}$, which is convergent in R. Then $\exists\ l \in R$ with $s_{k_n} \longrightarrow l$ as $n \longrightarrow \infty$.
This implies $s_n \longrightarrow l$ as $n \longrightarrow \infty$, showing the Cauchy sequence $\{s_n\}_{n=1}^{\infty}$ is convergent in R (with limit $l \in R$ here). Thus every Cauchy sequence in R is convergent in R.
Therefore R is a complete space.

Theorem

If $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence (or monotone) and it has a convergent subsequence $\{s_{k_n}\}_{n=1}^{\infty}$ with $s_{k_n} \longrightarrow l$ as $n \longrightarrow \infty$, then $s_n \longrightarrow l$ as $n \longrightarrow \infty$.

Proof: Let $\{s_n\}_{n=1}^{\infty}$ be a Cauchy sequence and it has a convergent subsequence $\{s_{k_n}\}_{n=1}^{\infty}$ and $\lim_{n\to\infty} s_{k_n} = l \in R$, for some l. As $\{s_n\}_{n=1}^{\infty}$ Cauchy, so $\forall\ \varepsilon > 0$, $\exists\ n_0 \in N$ such that $m, n \geq n_0 \Rightarrow |s_m - s_n| < \varepsilon/2$, and also ($\because k_n \geq n$),

$|s_m - s_{k_n}| < \varepsilon/2$. Taking $n \longrightarrow \infty$, ($k_n \geq n$), and using $s_{k_n} \longrightarrow l$ as $n \longrightarrow \infty$, we get $|s_m - l| < \varepsilon\ \forall\ m \geq n_0$.

This proves $\lim_{n\to\infty} s_n = l$.

Proof of above theorems, results are available in [1] – [4] or in any standard books on Real Analysis and also further details on topics.

Euclidean Spaces

The concept of a metric space was originated in the Ph. D. thesis of Maurice Frechet presented to University of Paris in 1906. The definition of metric presenly in use is given by German mathematician F. Hausdorff in 1914.

Metric and Metric Space: d is called a *metric* on a nonempty set X if d: $X \times X \longrightarrow R$ is a function such that

- $d(x, y) \geq 0\ \forall\ x, y \in X$ and $d(x, y) = 0$ iff $x = y$ (Positivity).
- $d(x, y) = d(y, x)\ \forall\ x, y \in X$ and (Symmetry)
- $d(x, y) \leq d(x, z) + d(z, y)\ \forall\ x, y, z \in X$. (Triangle Inequality) A set X with a metric d on it, is called a *metric space* and it is denoted by (X, d). Let $\{s_n\}$ be a sequence in a metric space (X, d). (a) If $l \in X$ is such that $\forall\ \varepsilon > 0$, $\exists\ n_0 \in \mathbf{N}$ with $n \geq n_0 \Rightarrow d(s_n, l) < \varepsilon$ then we say that $\{s_n\}$ is a *convergent sequence* and in this case we say that the sequence $\{s_n\}$ *converges to* l.

A sequence is said to be *divergent* if it is not convergent.
(b) Sequence $\{s_n\}_{n=1}^{\infty}$ is said to be *bounded* if $\exists$ fixed $M > 0$ such that $d(s_m, s_n) \leq M\ \forall\ m, n \in N$.

Results 1 If for any fixed $c \in X$, $\exists\ M > 0$ such that $d(s_n, c) \leq M\ \forall\ n \in N$, then the sequence $\{s_n\}_{n=1}^{\infty}$ in the metric space (X, d) is bounded.
This follows from triangle inequality: $d(s_m, s_n) \leq d(s_m, c) + d(s_n, c) \leq M + M = 2M\ \forall\ m, n \in N$. (c) Sequence $\{s_n\}$ is said to be a *Cauchy sequence* if $\forall\ \varepsilon > 0$, $\exists\ n_0 \in N$ such that
$m, n \geq n_0$ (or $m > n \geq n_0$) $\Rightarrow d((x_m, x_n) < \varepsilon$; or equivalently $n \geq n_0 \Rightarrow d(x_{n+p}, x_n) < \varepsilon\ \forall\ p \in N$.
Note that every convergent sequence is Cauchy, but in general its converse is not true.
Result 2. Every Cauchy sequence in a metric space is bounded.

Proof: Let $\{s_n\}_{n=1}^{\infty}$ be a Cauchy sequence in a metric space (X, d). Then for $\varepsilon = 1$, $\exists\ p \in N$ such that $m, n \geq p \Rightarrow d(s_m, s_n) < 1$. Now $L = \max\{d(s_i, s_j): 1 \leq i, j < p\} \in R$ and $L \geq 0$. For any $j < p$ and $n \geq p$, we have $d(s_j, s_n) \leq d(s_j, s_p) + d(s_p, s_n) < L + 1$. Then $d(s_m, s_n) < L + 1\ \forall\ m, n \in N$, showing that $\{s_n\}_{n=1}^{\infty}$ is a bounded sequence in X. (d) A metric space X is said to be a *complete space* if every Cauchy sequence in X is convergent in X. (e) $d(x, y) = |x - y|$, for $x, y \in R$, is a metric, called the *usual (Euclidean) metric* on R.

(f) For any $x = (x_1, x_2,..., x_n)$, $y = (y_1, y_2,..., y_n) \in R^m$; (where $m \in N$ fixed), $d(x, y) = \| \mathbf{x} - \mathbf{y} \| =$

$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}$ defines a metric on R^n, called the *usual* (or *Euclidean*) *metric* on R^m and under this metric R^m is called a *Euclidean space*.

Here $\| \mathbf{x} \| = \sqrt{x_1{}^2 + x_2{}^2 + \ldots + x_n{}^2}$ is called *norm* of x.

Note that $|\| \mathbf{x} \| - \| \mathbf{y} \|| \leq \| \mathbf{x} - \mathbf{y} \|$ $\forall$ x, y $\in R^m$.
A sequence $\{S_n\}$ in the Euclidean space R^n is *bounded* if $\exists$ a number $M > 0$ such that
$\| s_n \| = d(S_n, 0) \leq M \ \forall \ n \in N$. Here $0 = (0, 0, \ldots, 0) \in R^n$.

2.1 Lemma: if $S_n = (a_n, b_n, \ldots, t_n) \in R^m \ \forall \ n \in N$, then $\{S_n\}$ is a bounded sequence in R^m iff all sequences $\{a_n\}$, $\{b_n\}, \ldots, \{t_n\}$ are bounded in R.

Proof: Let $\{S_n\}$ be a bounded sequence in R^m. Then $\exists$ a number $M > 0$ such that

$\| s_n \| = \sqrt{a_n{}^2 + b_n{}^2 + \ldots + t_n{}^2} \leq M \ \forall \ n \in N$. $\Rightarrow |a_n| \leq M, |b_n| \leq M, \ldots, |t_n| \leq M \ \forall \ n \in N$. $\Rightarrow$ All m sequences $\{a_n\}$, $\{b_n\}, \ldots, \{t_n\}$ in R are bounded in R.

Conversely let all m sequences $\{a_n\}$, $\{b_n\}, \ldots, \{t_n\}$ in R are bounded in R.

Then $\exists$ a number $K > 0$, $(\frac{K}{\sqrt{m}} > 0)$, such that $|a_n| \leq \frac{K}{\sqrt{m}}$, $|b_n| \leq \frac{K}{\sqrt{m}}, \ldots, |t_n| \leq \frac{K}{\sqrt{m}} \ \forall \ n \in N$.

$\Rightarrow \| s_n \| = = \sqrt{a_n{}^2 + b_n{}^2 + \ldots + t_n{}^2} \leq \sqrt{(\frac{K}{\sqrt{m}})^2 + (\frac{K}{\sqrt{m}})^2 + \ldots + (\frac{K}{\sqrt{m}})^2} = K \ \forall \ n \in N$.

Hence $\{S_n\}$ is a bounded sequence in R^m.

Theorem
***Bolzano-Weierstrass Theorem*:** Every bounded sequence in a Euclidean space R^m has a convergent subsequence.

Proof: We prove the theorem using induction on m.

We have: Every bounded sequence in R has a convergent subsequence in R.

∴ Theorem follows for the Euclidean space R^m for m = 1.

We prove the theorem for m = 2: Let $\{s_n\}_{n=1}^{\infty}$ be any bounded sequence in R^2 and let

$s_n = (a_n, b_n) \ \forall \ n \in N$, i.e. $a_n, b_n \in R$. Then $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are bounded sequences in R.

Hence both sequences have convergent subsequences in R. Let $\{a_{k_n}\}_{n=1}^{\infty}$ be a convergent subsequence of $\{a_n\}_{n=1}^{\infty}$ and $l_1 \in$ R be its limit; i.e. $a_n \longrightarrow l_1$ as $n \longrightarrow \infty$.

Then $\{b_{k_n}\}_{n=1}^{\infty}$ is a subsequence of the bounded sequence $\{b_n\}_{n=1}^{\infty}$ and hence bounded in R and so it has a convergent subsequence in R, say $\{b_{k_{t_n}}\}_{n=1}^{\infty}$.

So $\exists$ $l_2 \in$ R such that $b_{k_{t_n}} \longrightarrow l_2$ as $n \longrightarrow \infty$. As any subsequence of a convergent sequence has the same limit that of the sequence, so the subsequence $\{a_{k_{t_n}}\}_{n=1}^{\infty}$ of the convergent sequence $\{a_{k_n}\}_{n=1}^{\infty}$ has the limit l_1. Thus $a_{k_{t_n}} \longrightarrow l_1$ as $n \longrightarrow \infty$.

We have $\left\{(a_{k_{t_n}}, b_{k_{t_n}})\right\}_{n=1}^{\infty}$ as a subsequence of the sequence $\{s_n\}_{n=1}^{\infty}$ in R^2 and $(l_1, l_2) \in R^2$, with $\lim_{n\to\infty} a_{k_{t_n}} = l_1$, $\lim_{n\to\infty} b_{k_{t_n}} = l_2$. Then for any $\varepsilon > 0$, $(\frac{\varepsilon}{\sqrt{2}} > 0)$, $\exists\ n_0 \in N$ such that

$$n \geq n_0 \Rightarrow |a_{k_{t_n}} - l_1| < \frac{\varepsilon}{\sqrt{2}} \text{ and } |b_{k_{t_n}} - l_2| < \frac{\varepsilon}{\sqrt{2}}$$

$$\Rightarrow d((a_{k_{t_n}}, b_{k_{t_n}}), (l_1, l_2)) = \sqrt{(a_{k_{t_n}} - l_1)^2 + (b_{k_{t_n}} - l_2)^2} < \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \varepsilon.$$

This proves that any bounded sequence $\{s_n\}_{n=1}^{\infty}$ in R^2 has a convergent subsequence $\left\{(a_{k_{t_n}}, b_{k_{t_n}})\right\}_{n=1}^{\infty}$ in R^2 (with $\lim_{n\to\infty} (a_{k_{t_n}}, b_{k_{t_n}}) = (l_1, l_2) \in R^2$).

Thus every bounded sequence in R^2 has a convergent subsequence in R^2, that is theorem is true for m = 2. Let m > 2 and assume that the theorem is true for m – 1, i.e. every bounded sequence in R^{m-1} has a convergent sequence in R^{m-1}. (Induction Hypothesis).

Let $\{S_n\}_{n=1}^{\infty}$ be any bounded sequence in R^2 and let $S_n = (A_n, b_n)\ \forall\ n \in N$, i.e. $A_n \in R^{m-1}$, $b_n \in R$. Then $\{A_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are bounded sequences in R^{m-1}, R respectively. Hence both sequences have convergent subsequences in R^{m-1} (by hypothesis), R respectively. Let $\left\{A_{k_n}\right\}_{n=1}^{\infty}$ be a convergent subsequence of $\{A_n\}_{n=1}^{\infty}$ and $l \in R^{m-1}$ be its limit; i.e. $A_n \longrightarrow l$ as $n \longrightarrow \infty$. Then $\left\{b_{k_n}\right\}_{n=1}^{\infty}$ is a subsequence of the bounded sequence $\{b_n\}_{n=1}^{\infty}$ and hence bounded in R and so it has a convergent subsequence in R, say $\left\{b_{k_{t_n}}\right\}_{n=1}^{\infty}$.

So $\exists\ l_2 \in R$ such that $b_{k_{t_n}} \longrightarrow l_2$ as $n \longrightarrow \infty$. As any subsequence of a convergent sequence has the same limit that of the sequence, so the subsequence $\left\{A_{k_{t_n}}\right\}_{n=1}^{\infty}$ of the convergent sequence $\left\{A_{k_n}\right\}_{n=1}^{\infty}$ has the limit l. Thus $A_{k_{t_n}} \longrightarrow l$ as $n \longrightarrow \infty$. We have $\left\{(A_{k_{t_n}}, b_{k_{t_n}})\right\}_{n=1}^{\infty}$ as a subsequence of the sequence $\{S_n\}_{n=1}^{\infty}$ in R^m and $(l, l_2) \in R^m$, with $\lim_{n\to\infty} A_{k_{t_n}} = l$, $\lim_{n\to\infty} b_{k_{t_n}} = l_2$. Then for any $\varepsilon > 0$, $(\frac{\varepsilon}{\sqrt{2}} > 0)$, $\exists\ n_0 \in N$ such that

$$n \geq n_0 \Rightarrow \| A_{k_{t_n}} - l \| < \frac{\varepsilon}{\sqrt{2}} \text{ and } |b_{k_{t_n}} - l_2| < \frac{\varepsilon}{\sqrt{2}}$$

$$\Rightarrow d((A_{k_{t_n}}, b_{k_{t_n}}), (l, l_2)) = \sqrt{\| A_{k_{t_n}} - l \|^2 + (b_{k_{t_n}} - l_2)^2} < \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \varepsilon.\ A_{k_{t_n}} - l$$

This proves that any bounded sequence $\{S_n\}_{n=1}^{\infty}$ in R^m has a convergent subsequence $\left\{(A_{k_{t_n}}, b_{k_{t_n}})\right\}_{n=1}^{\infty}$ in R^m (with $(A_{k_{t_n}}, b_{k_{t_n}}) = (l, l_2) \in R^m$).

Hence the theorem follows by the first principle of mathematical induction.

Lemma: Every Cauchy sequence in a Euclidean space is bounded.

Proof: Let $\{s_n\}_{n=1}^{\infty}$ be any Cauchy sequence in a Euclidean space R^m.

Then for $\varepsilon = 1$, $\exists\ p \in N$ such that $k, n \geq p \Rightarrow \|s_n - s_k\| < 1 \Rightarrow \| s_n \| - \| s_p \| < 1$, I.e.
$\| s_n \| < \| s_p \| + 1\ \forall\ n \geq p$.

Let M = max {$\| s_1 \|$, $\| s_2 \|$,..., $\| s_{p-1} \|$, $\| s_p \|$ + 1}.

Then M ∈ R, M > 0 and $\| s_n \| \le M$ ∀ n ∈ N. Hence the Caucy sequence $\{s_n\}_{n=1}^{\infty}$ is bounded.

Theorem: Every Euclidean space is complete, i.e. for each m ∈ **N**, $\mathbf{R}^m$ is a complete metric space under the Euclidean metric.

Proof: Let $\{s_n\}_{n=1}^{\infty}$ be any Cauchy sequence in R^m. Then $\{s_n\}_{n=1}^{\infty}$ is a bounded sequence and hence it has a convergent subsequence, say $\{s_{k_n}\}_{n=1}^{\infty}$ and so $s_{k_n} \longrightarrow l$ as n ⟶ ∞ for some $l \in R^m$. Now $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence, so ∀ ε > 0, (ε/2 > 0), ∃ n_0 ∈ N such that t, n ≥ n_0 ⇒ $\| s_t - s_n \| < \varepsilon/2$ (E1).

As $k_n \ge n$ ∀ n ∈ N, since $\{s_{k_n}\}_{n=1}^{\infty}$ is a subsequence of the sequence $\{s_n\}_{n=1}^{\infty}$, we have by (E1)
n, t ≥ n_0 ⇒ $\| s_t - s_{k_n} \| < \varepsilon/2$. (E2).

Taking n ⟶ ∞ and using $s_{k_n} \longrightarrow l$, we have by (E2),
t ≥ n_0 ⇒ $\| s_t - l \| \le \varepsilon/2$.

Thus ∀ ε > 0, ∃ n_0 ∈ N such that n ≥ n_0 ⇒ $\| s_n - l \| < \varepsilon$, i.e. $\lim_{n\to\infty} s_n = l$.
Hence $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence in R^m with limit $l \in R^m$.

Conclusions

The set of complex numbers C is identified with R^2 where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ ∈ C are identified with $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2) \in R^2$. Usual metric on C is the Euclidean metric on R^2, since $|z_1 - z_2| = |(x_1 - x_1) + i(y_1 - y_2)| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d((x_1, y_1), (x_2, y_2))$.

Since R^2 is complete, so C is complete. As every bounded sequence in R^2 has a convergent subsequence in R^2, so every bounded sequence in C has a convergent subsequence in C.
Also, for any m ∈ N, R^{2m} is a complete space, so the unitary space C^m (identified with R^{2m}) is complete. Every bounded sequence in R^{2m} has a convergent subsequence, so every bounded sequence in C^m has a convergent subsequence (BW Theorem).

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