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Derivative-Free Conformable Iterative Methods for Solving Nonlinear Equations

[1]Sanjay Singh and [2]Pratima Ojha

[1, 2]Department of Mathematics, Madhyanchal Professional University, Bhopal, Madhya Pradesh, India

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Corresponding Author: Sanjay Singh

Abstract

The iterative solving nonlinear algebraic or transcendental equations by use of conformable derivative approximations. For the purpose of solving nonlinear equations, we modify the approximation of conformable derivatives and build iterative schemes (such as Secant-type techniques and Steffensen-type schemes) that do not need conformable derivatives. Our sources tell us that the Secant conformable scheme and the Steffensen conformable technique are the first two optimum conformable derivative-free schemes. The Secant scheme is a memory process. All the while maintaining the classical case order, a convergence analysis is performed, and numerical performance is verified to verify that the theoretical findings are valid. Utilizing these techniques in tandem with extensive datasets of convergent initial estimates has the potential to provide numerical benefits compared to more traditional approaches. Conformable calculus offers an alternative to fractional calculus, which does not evaluate particular functions like Gamma or Mittag-Leffler functions, and has a minimal processing cost. We now look at the schemes presented in this book and how they rely on the initial estimations, particularly with regard to the convergence planes.

Keywords: Derivative, Iterative, Nonlinear, Equations and numerical

Introduction

Many theoretical and applied problems require us to find the solution $\bar{x} \in R$ of the nonlinear equation $f(x) = 0$, where $f: I \subseteq R \rightarrow R$. Due to most of the nonlinear problems not having an analytical solution, many authors have proposed numerical methods by means of fixed-point schemes to approximate the solution of $\bar{x}$. Most of these procedures possess the evaluation of an integer order derivative, or its approximation. In the recent literature, some authors have introduced several numerical methods with no integer order derivatives (fractal derivative, fractional derivative, and conformable derivative). These derivatives of order $\alpha \in (0, 1]$ establish a generalization of a classical one, which is a particular case when the order is $\alpha = 1$. Nonintegral derivatives can be used to model many applied problems because of the higher degree of freedom of its tools compared to classical calculus tools. the first Newton's methods with fractal derivatives are presented, whose order of convergence is quadratic.

Iterative methods with fractional derivatives do not hold the order of convergence of their classical versions; they need higher-order fractional derivatives to increase the order of convergence, preventing it from being possible to obtain optimal order procedures according to Kung and Traub conjecture unlike schemes with conformable derivatives. So, another approach is the conformable calculus whose low computational cost constitutes an advantage versus fractional calculus, due to special functions as Gamma or Mittag–Leffler functions are not evaluated. In that sense, several conformable iterative schemes were designed the scalar and vectorial versions of a Newton-type method with conformable derivative/Jacobian, respectively, and a general technique is designed in order to obtain the conformable version of any scalar classical procedure. Also, the proposed the first multipoint conformable method for solving nonlinear systems (a Traub-type method).

Finally, some derivative-free schemes were designed (a Steffensen-type and Secant-type procedures) with an approximation of conformable derivatives and a Traub–Steffensen-type (in scalar and vectorial version). The theoretical convergence order of these methods is preserved in practice. Indeed, these methods show good qualitative behavior, improving even their respective classical cases in some numerical aspects. Most of these fractal, fractional, and conformable schemes mentioned above need the evaluation of fractal, fractional, or conformable derivatives, respectively. Since conformable procedures have presented many advantages versus fractional ones, in this manuscript, we focus in the approximation of conformable derivatives in order to design, to our knowledge, the first conformable derivative-free iterative methods to solve nonlinear equations: a Steffensen-type method and a Secant-type method we also compare them with their classical partners.

Literature Review

Damien Scieur (2017) [1] We show that the gradient flow equation can be solved using multi-step integration schemes obtained from numerical analysis, and that accelerated optimization methods may be seen as particular instances of these schemes. A family of multi-step methods, including accelerated algorithms, is constructed using classical conditions from numerical analysis; the fundamental gradient flow differential equation is studied. This seems lethargic in comparison to what has happened recently. One possible intuitive explanation for acceleration events is to use multi-step methods to integrate the differential equation with bigger step sizes.

Mustafa Turkyilmazoglu (2017) [2] The traditional Adomian decomposition approach is discussed as a method for solving integra-differential, ordinary or partial type linear, nonlinear algebraic, or problems. The updated, more refined method makes use of an internal parameter known as the convergence control parameter. It is possible to regulate the method's convergence and the pace of convergence with this parameter. Creating constant level curves to aid in value identification is the first step in establishing a strategy to effectively attain the appropriate convergence control parameter. The squared residual error of the problem under consideration is the basis of this method. In cases when the conventional Adomian decomposition method is unsuccessful, it has been demonstrated that the optimal Adomian decomposition method can converge to the correct answer. It has been demonstrated that the current strategy can potentially speed up the convergence rate when both methods converge. In addition, it has been demonstrated that, in comparison to the standard Adomian method, the optimal Adomian decomposition approach has the potential to significantly enlarge the narrow interval of the convergent physical solution's limited domain.

Mojtaba Hajipour (2018) [3] An accurate discretization strategy for addressing one-, two-, and three-dimensional extremely nonlinear Bratu type problems is studied in this research. The issue that is being studied may be reduced to the solution of a highly nonlinear algebraic system by the discretization of the nonlinear equation through the use of a fourth-order nonstandard compact finite difference formula. It is necessary to make use of a modified nonlinear solver in order to solve the resulting nonlinear system. In order to locate the bottom and higher branches of the Bratu's issue, the new method is precise, quick, uncomplicated, and extremely efficient. A comparison of the results of numerical simulations and comparative analyses for one-, two-, and three-dimensional instances demonstrates that the new approach is not only simpler to implement but also more accurate than the other methods that are currently available in the literature.

Rizk M. Rizk-Allah (2021) [4] In order to solve broad systems of nonlinear equations, this work proposes Q-SCA, a quantum-based sine cosine method. The Q-SCA method is an amalgamation of the merits of the QLS and SCA algorithms, which stand as a consequence of their merger. By increasing the variety of choices, it hopes to escape the trap of settling for locally optimal solutions. Two of the most important aspects of the proposed Q-SCA are the acceleration of the optimal searching process and the convergence property. There are two ways that the suggested Q-SCA can function. It is an enhanced version of SCA that, first and foremost, searches for solutions by dynamically modifying the space surrounding the target solution. This is one method to gradually narrow the search space once the optimal solution is located. To top it all off, bidirectional equations are utilized in an innovative way to update the answers. The results of the SCA stage are then refined with the help of QLS. By using this approach, the suggested Q-SCA is able to achieve high levels of exploration and exploitation and converges to high-quality solutions precisely and gradually. Twelve separate sets of nonlinear equations and two separate electrical applications are used to evaluate the suggested method. To further guarantee its scalability, the proposed Q-SCA method is tried out on costly, large-scale challenges such as the CEC 2017 benchmark and realistic optimum power dispatch (OPD).

Francesco Biral (2016) [5] New developments in theory, algorithms, and computer power have made it possible to solve complicated control problems that are well-suited for offline and online industrial applications. The first part of this book provides a technical analysis of the various problem-solving approaches that fall into three main groups: direct techniques, indirect methods, and dynamic programming. The available software is briefly summarized, and the advantages and disadvantages of each technique are discussed. To back up this subject, we offer a concrete example. Specifically, we find that the direct technique, which uses interior point methods and non-linear programming solvers, outperforms the indirect method numerically. The study's indirect method, founded on the Pontryagin Minimum Principle (PMP), is presented in Section 2. Optimal control issues arise in many different contexts, and this package includes PINS, a suite of software tools that may help you define the problem, write C++ code automatically, and find numerical solutions to these problems. In conjunction with implementing PMP, optimal conditions may be obtained analytically.

Research Methodology

Following is the iterative methodology for the improved NR method:

$$z_{n+1} = z_n - \frac{f(z_n) f'(z_n)}{\alpha_0 (f'(z_n))^2 + \alpha_1 f(z_n) f''(z_n)},$$

where α_0 and α_1 genuine parameters that are free.

This being the case, $\alpha_0 = 1$ and $\alpha_1 = 0$ simplified from its iterative form (1) to

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)},$$

This is the traditional approach to NR, but when $\alpha_0 = 1$ and $\alpha_1 = -1/2$ it lowers to

$$z_{n+1} = z_n - \frac{2f(z_n)\, f'(z_n)}{2(f'(z_n))^2 - f(z_n)\, f''(z_n)},$$

and that is the Halley's approach.

We want to clarify how α_1 affects the behavior of the improved NR technique. Simplifying without sacrificing generalizability, we sought out to $\alpha_0 = 1$. The function that is polynomial f (z) = $z^7 - 1$ serves as an example for testing purposes. See Figure 1. for a representation of the circle with a radius of 1 that contains all seven complex conjugate roots and one real root of the equation f(z) = 0.

The following is how the iterative technique operates: A starting condition on the complex plane expressed as a complex number is first provided $z = R + iI$. In fact, the algorithm is activated by this beginning condition, and iterations are continued until we achieve the root of the polynomial function, naturally with the specified precision.

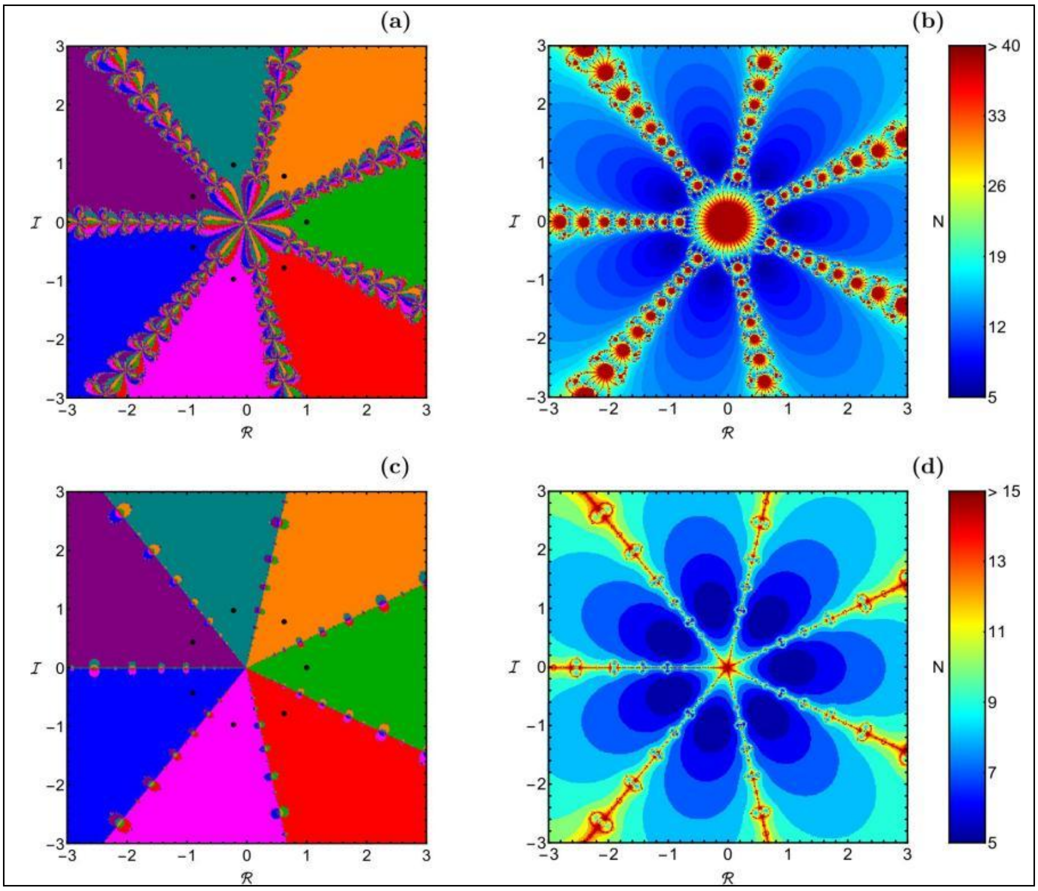


Fig 1: Color diagrams of the basins for the traditional Newton-Raphson technique (a) on the top left and the Halley approach (c) on the bottom left. Black dots indicate where each of the seven roots is located. The seven roots are represented by the colors, which are: R1 (green); R_2 (orange); R_3 (teal); R_4 (purple); R_5 (blue); R_6 (magenta); R_7 (red). (b) and (d): How many iterations are necessary and their distributions.

Data Analysis

It is not possible to obtain optimal order procedures according to Kung and Traub conjecture using iterative methods with fractional derivatives; unlike schemes with conformable derivatives, these methods require higher-order fractional derivatives to increase the order of convergence, which is not possible with iterative methods with fractional derivatives. As an alternative to fractional calculus, which does not evaluate specific functions like Gamma or Mittag-Leffler functions, there is the conformable calculus, which has a low processing cost. So, many conformable iterative methods were developed, including

the scalar and vectorial variants of a Newton-type method with conformable derivative/Jacobian, and a generic way to derive the conformable variant of any scalar classical process.
In this manuscript, we focus on the approximation of conformable derivatives to design the first conformable derivative-free iterative methods to solve nonlinear equations: a Secant-type method and a Steffensen-type method. We also compare them with their classical counterparts, since conformable procedures have many advantages over fractional ones.
A few definitions from conformable calculus that are useful to keep in mind are: It is possible to define the left conformable derivative of a function f: [a, ∞) → R, where a, α, and x are all elements of R and x > a, as follows: beginning with a and having an order α ∈ (0, 1].

$$(T_\alpha^a f)(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon(x-a)^{1-\alpha}) - f(x)}{\varepsilon}.$$

The function f is α-differentiable if and only if this limit exists. Assuming f is differentiable, we may say that (a) ($T_\propto^a$ f) (x) = f ′(x) $(x - a)^{1-\alpha}$. Assuming f is α-differentiable in the interval (a, b) for some b ∈ R, $(T_\alpha^a f)(a) = \lim_{x\to a^+}(T_\alpha^a f)(x).$

The characteristic of non-fractional derivatives is maintained by this derivative: $T_\propto^a$ K= 0, in which K is a fixed value. It is not necessary to evaluate any particular function for this kind of derivative, as previously stated.

The following outcome demonstrates the provision of a suitable conformable Taylor series
Theorem 1: Assume that f(x) is a conformable function whose derivatives begin at a and that it is endlessly α-differentiable with respect to α (0, 1]. Next, we may get the conformable Taylor series of f(x) by

$$f(x) = f(a_1) + \frac{(T_\alpha^a f)(a_1)}{\alpha}\delta_1 + \frac{(T_\alpha^a f)^{(2)}(a_1)}{2\alpha^2}\delta_2 + R_2(x, a_1, a),$$

being $L = a_1 - a$, $H = x - a$, $\delta_1 = H^\propto - L^\propto$, $\delta_2 = H^{2\propto} - L^{2\propto} - 2L^{2\propto}\delta_1$,....
It is plain to see that $\delta_2 = \delta_1^2$, $\delta_3 = \delta_1^3$, on and on. Hence, the expression for (4.41) is

$$f(x) = f(a_1) + \frac{1}{\alpha}(T_\alpha^a f)(a_1)\delta_1 + \frac{1}{2\alpha^2}(T_\alpha^a f)^{(2)}(a_1)\delta_1^2 + R_2(x, a_1, a).$$

We must provide a generalization of the order of convergence (The R-order) since our proposed Secant-type technique incorporates memory. However, before we do so, let's examine the idea of the R-factor:

Definition 2: Let φ be an iterative approach that converges to a certain limit β, and allow $\{x_k\}$ constitute a random order of in $\mathbb{R}^n$ approaching β. Afterwards, the sequence's R-factor $\{x_k\}$ is

$$R_m(x) = \begin{cases} \limsup\limits_{k\to\infty} ||x_k - \beta||^{1/k}, & \text{for } m = 1, \\ \limsup\limits_{k\to\infty} ||x_k - \beta||^{1/m^k}, & \text{for } m > 1. \end{cases}$$

We are now able to determine the R-order.

Definition 3: How fast an iterative algorithm converges, in R-order φ at the point β is

$$O_R(\phi, \beta) = \begin{cases} +\infty, & \text{if } R_m(\phi, \beta) = 0 \ \ \forall m \in [1, +\infty), \\ \inf\{m \in [1, +\infty) : R_m(\phi, \beta) = 1\}, & \text{in other case.} \end{cases}$$

The R-order of a memory-enabled iterative technique is related to the roots of a characteristic polynomial, as stated in the following conclusion:

Theorem 4: Let φ cycle through a set of steps, with each step generated by memory $\{x_k\}$ of estimates for the base $\bar{x}$, Moreover, what if the sequence $\{x_k\}$ merges with $\bar{x}$. Assuming a non-zero constant η, non-negative integers and t_i, i ∈ [0, m], to the point that

$$|e_{k+1}| \leq \eta \prod_{i=0}^{m} |e_{k-i}|^{t_i},$$

When satisfied, the iterative scheme's R-order φ satisfies

$$O_R(\phi, \bar{x}) \geq s^*,$$

being s∗ the one and only positive polynomial

$$s^{m+1} - \sum_{i=0}^{m} t_i s^{m-i} = 0.$$

As a last consideration, the asymptotical error constant C of an iteration function φ(x) of order p is defined as

$$C = \lim_{x \to \bar{x}} \frac{\bar{x} - \phi(x)}{(\bar{x} - x)^p},$$

Knowing the error constant of another iterative technique of the same order, the following result allows one to calculate the error constant of an iterative scheme with p-order of convergence.

Inferring the Procedures
The following conformable finite division difference of linear order is seen as an approximation:

$$(T_\alpha^a f)(x) \approx \frac{f(x + \varepsilon(x - a)^{1-\alpha}) - f(x)}{\varepsilon}, \quad \varepsilon \neq 0.$$

In all cases, whether the processes are scalar or vectorial, one-point or multipoint, the conformable schemes maintain the theoretical order of their classical forms (where α = 1). It is worth considering if the conformable (memory or non-memory) derivative-free approach maintains the same order of convergence as its classical counterpart. We do this by using the suggested universal approach, which can be used to determine the conformable partner of any known process, and proving that these procedures maintain the same order of convergence as their classical counterparts.

According to the standard procedure, the classical approach

$$\varphi(x) = x - g(x)\, f(x),$$

contains the version that is easily modified

$$\varphi(x) = a + ((x - a)^{\alpha} - \alpha g_{\alpha}(x)\, f(x))^{1/\alpha}.$$

If the classical derivatives off (x) are included in g(x) in then g_α(x) the formable derivatives off(x).

Finding the analytical formulation of g(x) is necessary to acquire its conformable version from a classical scheme.
Regarding the method used by Steffensen:

$$\phi_1(x) = x - \frac{f(x)}{f(x + f(x)) - f(x)} f(x),$$

where $\frac{f(x+f(x))-f(x)}{f(x)} \neq 0$ approaches the classical derivative approximately. So,

$$g(x)=\frac{f(x)}{f(x+f(x))-f(x)}.$$

When it comes to,

$$g_\alpha(x)=\frac{f(x)}{f(x+f(x)(x-a)^{1-\alpha})-f(x)},$$

Analysis of Convergence

The following finding defines the parameters under which SeCO's quadratic order of convergence holds. We make use of the symbols $x = x_k$ and $\varphi(x) = x_{k+1}$.

Theorem 5: First, we'll look at a differentiable function that meets our requirements $f : I \subseteq \mathbb{R} \to \mathbb{R}$ When there is a zero on the open interval I $\bar{x}$ off (x). Let us pretend for a moment that x_0, our first estimate, is near enough to $\bar{x}$. Convergence in conformable Steffensen's scheme (SeCO), which is defined by

$$x_{k+1}=a+\left((x_k-a)^\alpha-\alpha\frac{(f(x_k))^2}{f(x_k+f(x_k)(x_k-a)^{1-\alpha})-f(x_k)}\right)^{1/\alpha}, \quad k=0,1,2,\ldots,$$

equals or exceeds 2, α ∈ (0, 1], it has an error equation that is

$$e_{k+1}=\left(\left(1+f'(\bar{x})(\bar{x}-a)^{1-\alpha}\right)C_2+\frac{1}{2}\frac{1-\alpha}{\bar{x}-a}\right)e_k^2+O\left(e_k^3\right),$$

where $C_q=\frac{f^{(q)}(\bar{x})}{q!f'(\bar{x})}, q\geq 2,$ to the point that $x_k > a, k = 0, 1, 2\ldots.$

Proof: Being aware which $x_k = e_k + \bar{x}$, the extended Taylor series of f (x_k) and [f $(x_k)]^2$ about $\bar{x}$ are

$$f(x_k)=f'(\bar{x})\left[e_k+C_2e_k^2+C_3e_k^3\right]+O\left(e_k^4\right),$$

And

$$[f(x_k)]^2=[f'(\bar{x})]^2\left[e_k^2+2C_2e_k^3\right]+O\left(e_k^4\right),$$

that is, being apart $C_q=\frac{f^{(q)}(\bar{x})}{q!f'(\bar{x})}$, for $q\geq 2$.

Recall that the extended binomial theorem is stated

$$(x+y)^r=\sum_{k=0}^{\infty}\binom{r}{k}x^{r-k}y^k, \quad k\in\{0\}\cup\mathbb{N},$$

$$\binom{r}{k} = \frac{\Gamma(r+1)}{k!\Gamma(r-k+1)}, \quad k \in \{0\} \cup \mathbb{N},$$

Numerical Results

We utilized MATLAB R2020a with double precision arithmetic to get the findings given here $|x_{k+1} - x_k| < 10^{-8}$ or $|f(x_{k+1})| < 10^{-8}$ as a cutoff for iterations, with no more than 500 allowed. Approximately how quickly computations converge (ACOC)

$$ACOC = \rho = \frac{\ln(|x_{k+1} - x_k| / |x_k - x_{k-1}|)}{\ln(|x_k - x_{k-1}| / |x_{k-1} - x_{k-2}|)}, \quad k = 2, 3, 4, \ldots,$$

utilized to verify that the theoretical convergence order is likewise retained in reality.

We now compare each scheme with its traditional counterpart (where $\propto = 1$) and test six nonlinear functions using the methodologies we defined in the preceding section. Our selection for EeCO is $x_{-1} = x_0 + 1$ before running the first cycle, we make a =-10 for all approaches, and α (0, 1].

For every test function, we display the outcomes from the two techniques discussed before (SeCO and EeCO) in each table; in both cases, x_0 is the same.

The primary purpose of the exam is $f_1(x) = -12.84x^6 - 25.6x^5 + 16.55x^4 - 2.21x^3 + 26.71x^2 - 4.29x - 15.21,$ having deep and meaningful origins

$$\bar{x}_1 \approx 0.82366 + 0.24769i,\ \bar{x}_2 \approx 0.82366 - 0.24769i,$$
$$\bar{x}_3 \approx -2.62297,\ \bar{x}_4 \approx -0.584,\ \bar{x}_5 \approx -0.21705 + 0.99911i, \text{ and } \bar{x}_6 \approx -0.21705 - 0.99911i.$$

Table 1. shows that when $\propto = 1$, the number of repetitions needed by SeCO is equal to or slightly higher than that of the conventional Steffensen's technique, and that ρ may be somewhat more than 2 $\propto \neq 1$. It should be noted that SeCO requires an initial estimate x_0 that is very close to $\bar{x}_4$ in order to merge with any $\propto$.

We observe that EeCO require in some cases less iterations than Secant scheme for most values of $\propto$, and the ACOC can be slightly higher than 1.618.

Table 1: Results for f1(x), with preliminary estimations x0 = −0.58 for SeCO, and x−1 = 0.42 and x0 = −0.58 for EeCO

	SeCO Method					EeCO Method				
α	$\bar{x}$	$f_1(x_{k+1})$	$x_{k+1} - x_k$	Iter	ρ	$\bar{x}$	$f_1(x_{k+1})$	$x_{k+1} - x_k$	Iter	ρ
1	$\bar{x}_4$	1.07×10^{-14}	2.18×10^{-10}	5	2.00	$\bar{x}_4$	9.95×10^{-14}	3.48×10^{-10}	5	2.38
0.9	$\bar{x}_4$	2.06×10^{-13}	4.04×10^{-9}	5	2.01	$\bar{x}_4$	3.60×10^{-10}	6.53×10^{-8}	5	2.47
0.8	$\bar{x}_4$	3.48×10^{-11}	6.31×10^{-8}	5	2.03	$\bar{x}_4$	1.50×10^{-10}	3.71×10^{-8}	5	2.23
0.7	$\bar{x}_4$	6.78×10^{-9}	7.87×10^{-7}	5	2.07	$\bar{x}_4$	6.02×10^{-12}	6.64×10^{-9}	4	1.48
0.6	$\bar{x}_4$	2.04×10^{-12}	1.22×10^{-8}	6	2.02	$\bar{x}_4$	1.88×10^{-9}	3.22×10^{-7}	4	0.90
0.5	$\bar{x}_4$	6.98×10^{-9}	6.37×10^{-7}	6	2.07	$\bar{x}_4$	3.39×10^{-9}	4.64×10^{-7}	4	0.80
0.4	$\bar{x}_4$	9.73×10^{-11}	6.72×10^{-8}	7	2.04	$\bar{x}_4$	5.25×10^{-9}	5.85×10^{-7}	4	0.74
0.3	$\bar{x}_4$	5.16×10^{-12}	1.39×10^{-8}	8	2.02	$\bar{x}_4$	8.13×10^{-9}	7.30×10^{-7}	4	0.70
0.2	$\bar{x}_4$	6.36×10^{-13}	4.01×10^{-9}	9	2.02	$\bar{x}_4$	2.24×10^{-13}	2.09×10^{-10}	5	2.53
0.1	$\bar{x}_4$	2.16×10^{-11}	2.24×10^{-8}	9	2.03	$\bar{x}_4$	6.54×10^{-13}	3.29×10^{-10}	5	2.63

We also have a test function that is $f_2(x) = \sin x - x2 + 1$, is rooted in reality $\bar{x}_1 \approx -0.6367$ and $\bar{x}_2 \approx 1.4096$.

Table 2. shows that SeCO converges more quickly than its classical counterpart and finds a different root. However, when the number of iterations exceeds 500, no results are displayed and the method converges to a point that is not a root of $f_2(x)$ because one of the stopping criteria is significantly larger than zero. We see that EeCO may need the same number of repetitions as its classical counterpart, and ρ can be marginally more than 1.618.

Table 2: Results for f2(x), with preliminary estimations $x_0 = 2$ for SeCO, and $x_{-1} = 3$ and $x_0 = 2$ for EeCO.

SeCO Method						EeCO Method				
α	$\bar{x}$	$f_2(x_{k+1})$	$x_{k+1} - x_k$	Iter	ρ	$\bar{x}$	$f_2(x_{k+1})$	$x_{k+1} - x_k$	Iter	ρ
1	$\bar{x}_2$	6.30×10^{-12}	1.60×10^{-6}	6	2.01	$\bar{x}_2$	1.26×10^{-10}	5.21×10^{-7}	6	1.62
0.9	$\bar{x}_1$	2.93×10^{-14}	1.20×10^{-7}	5	2.00	$\bar{x}_2$	1.59×10^{-9}	2.28×10^{-6}	6	1.62
0.8	-	-	-	-	-	$\bar{x}_2$	1.47×10^{-13}	6.57×10^{-7}	7	1.63
0.7	-	-	-	>500	-	$\bar{x}_2$	6.32×10^{-12}	6.13×10^{-8}	7	1.63
0.6	-	-	-	>500	-	$\bar{x}_2$	2.04×10^{-10}	4.78×10^{-7}	7	1.64
0.5	-	-	-	>500	-	$\bar{x}_2$	4.93×10^{-9}	3.11×10^{-6}	7	1.65
0.4	-	-	-	>500	-	$\bar{x}_2$	3.84×10^{-12}	3.46×10^{-8}	8	1.61
0.3	-	-	-	>500	-	$\bar{x}_2$	3.46×10^{-10}	5.10×10^{-7}	8	1.60
0.2	-	-	-	>500	-	$\bar{x}_2$	3.97×10^{-13}	6.88×10^{-9}	9	1.64
0.1	-	-	-	>500	-	$\bar{x}_2$	9.16×10^{-11}	1.82×10^{-7}	9	1.65

The following function is a test $f_3(x) = \left(\sin x - \frac{x}{2}\right)^2$, involving two distinct roots 1.8956, $\bar{x}_2 \approx 0$, and $\bar{x}_3 \approx 1.8956$.

Qualitative Performance

The dependence on the starting estimates of the schemes proposed in this book is now examined, specifically with respect to convergence planes. For both methods, the starting estimate x_0 is shown on the abscissa axis, and the ordinate axis is the same as the derivative's order α. The number of points that may be displayed using various colors is defined at 400. The pairs (x_0, α) that converge to one of the roots with a tolerance of 10^{-3} are indicated by those that are not black. There are a variety of colors linked to certain roots. Thus, if the iterative procedure does not converge to a root after 500 iterations, the point will be displayed in black.

We determine the proportion of convergent (x_0, α) for each convergence plane in order to compare how efficient these approaches are. In every plane, x_0 [5, 5], and each procedure, we set a = 10 and α (0, 1]. Regarding EeCO, we opt for $x_1 = x_0 + 1$ to carry out the first iteration in the same manner as the numerical tests.

As shown in Figure 2., although EeCO reaches about 43% convergence, all the roots are acquired, but SeCO only reaches about 0.2% convergence and obtains one root.

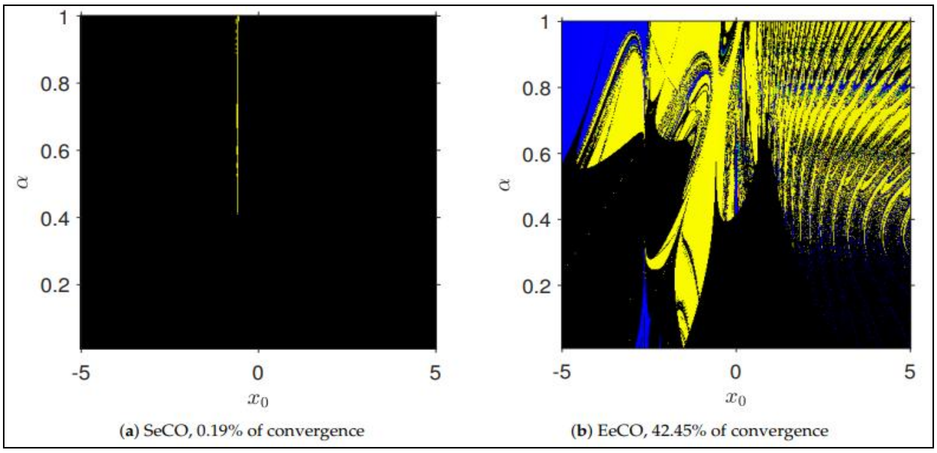


Fig 2: Alignment axes for $f_1(x)$

As shown in Figure 3, over 67% of convergence is achieved by SeCO and nearly 97% by EeCO. On each plane, we get the two roots.

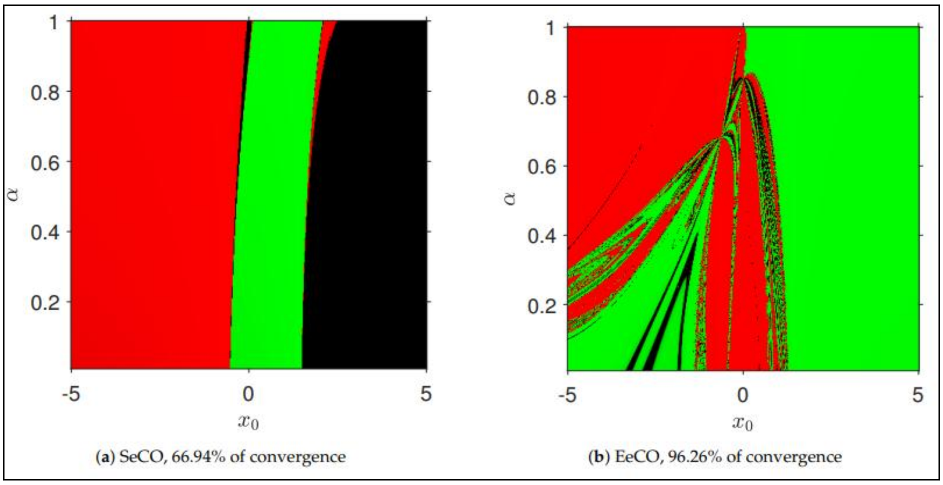
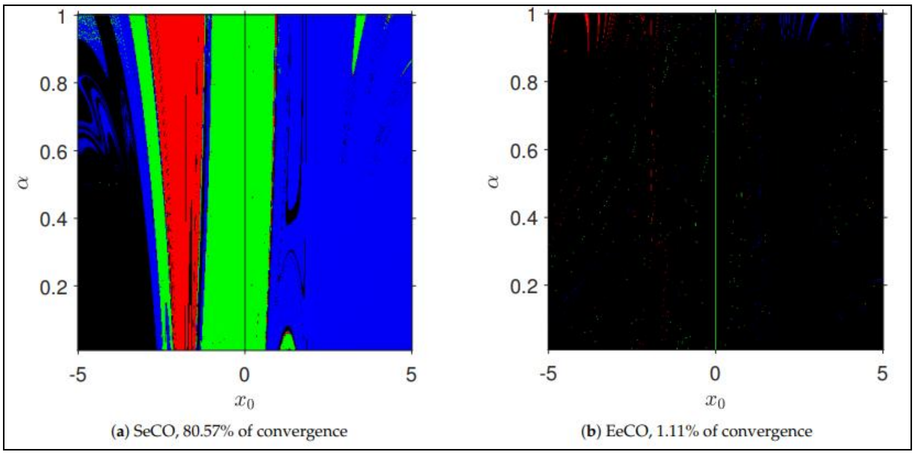


Fig 3: Alignment axes for $f_2(x)$.

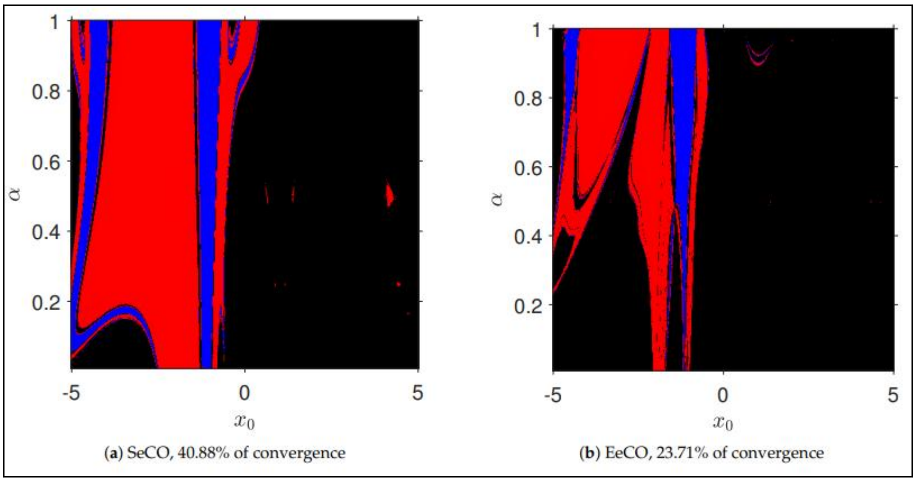
As seen in Figure 4., whereas EeCO only manages to obtain around 1% convergence, SeCO manages to accomplish about 81%. The three origins may be located.



(a) SeCO, 80.57% of convergence (b) EeCO, 1.11% of convergence

Fig 4: Convergence planes for $f_3(x)$.

Figure 5. shows that convergence is achieved by SeCO at about 41% and by EeCO at around 24%. On each plane, we get the two roots.



(a) SeCO, 40.88% of convergence (b) EeCO, 23.71% of convergence

Fig 5: Convergence planes for $f_4(x)$.

Conclusion

In the modified secant method for non-linear equation systems, develop hybrid approaches that use it in conjunction with other numerical techniques, and get derivative approximations of higher order. Both the first conformable scheme with memory (EeCO) and the optimal derivative-free iterative methodology (SeCO) were developed. To address nonlinear equations, we revise the conformable derivative approximation and construct iterative schemes (e.g., Secant-type approaches and Steffensen-type schemes) that are not dependent on conformable derivatives. Two of the most conformable derivative-free schemes, according to our sources, are the Secant conformable scheme and the Steffensen conformable approach. A low-processing-cost alternative to fractional calculus is the conformable calculus, which evaluates specified functions such as Gamma or Mittag-Leffler functions. The scalar and vectorial variants of a Newton-type method with conformable derivative/Jacobian were created as iterative schemes with conformable parameters, and a generic approach was devised to derive the conformable variant of any scalar classical process.

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